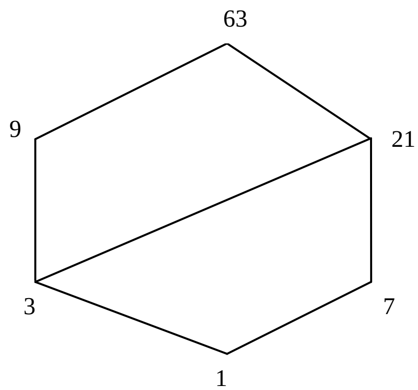
Q.1 a. Show that $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q$.

Answer:

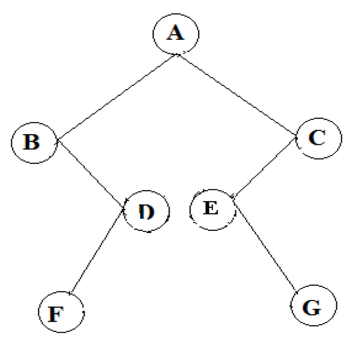
p	q	~p	p ∧ q	~p v q	~(p ∧ q)	A	B	A v B
F	F	T	F	T	T	F	F	F
F	T	T	F	T	T	F	T	T
T	F	F	F	F	T	F	F	F
T	T	F	T	T	F	T	F	T

b. Let $(D_{63}, \leq)$ be a lattice of all positive divisors of 63 and $x \leq y$ means x divides y. Draw the Hasse diagram.

Answer: D_{63} is the set of all positive divisors of 63, so $D_{63} = \{1, 3, 7, 9, 21, 63\}$. Thus the Hasse diagram of $(D_{63}, \leq)$ is shown below



c. Write the pre-order, post-order and in-order traversal for the given tree.



Answer: Pre-Order: ABDFCEG
In-Order: BFDAEGC
Post-Order: FDBGECA

d. Given the transition table of a finite machine M, q_1 is the final state.

Input→ State↓	0	1
$\to q_0$	q_1	q_0
q_1	q_1	q_2
q_2	q_0	q_1

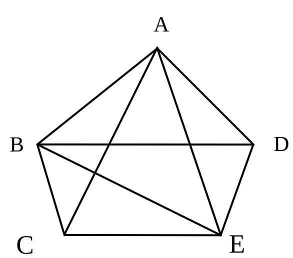
Show the processing of string 10101. Is it accepted by the machine?

Answer:
The processing of the string w = 10101 is as follows:

$\to q_0 10101$ |---- $1q_0 0101$ |---- $10q_1 101$ |---- $101q_2 01$ |--- $1010q_0 1$ |--- $10101q_0$

We know that a string is said to be accepted by a machine if after processing the string the machine reaches its final state. But here after processing the string machine goes into state q_0 which is not the final state of M, hence w = 10101 is not accepted by machine M.

e. Define path. Find all paths of length 3 between vertex A and C in the given graph G,



Answer: Path: A path is an alternating sequence of vertices and edges such that no vertex and edge appears more than once in the sequence.
The total no of edges gives the length of the path.
Thus the paths of length 4 between vertex A and C are:

1. A---B---D---E---C
2. A---D---B---E---C
3. A---D---E---B---C

f. Prove that A – (B ∩ C) = (A – B) ∩ (A – C).

Answer:
Let $x \in A-(B \cap C)$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$
$\Rightarrow x \in A$, $x \notin B$ and $x \notin C$
$\Rightarrow x \in A$, $x \notin B$ and $x \in A$ and $x \notin C$

$\Longrightarrow x \in (A - B)$ and $x \in (A - C)$
$\Longrightarrow x \in (A - B) \cap (A - C)$
Thus $A - (B \cap C) \subseteq (A - B) \cap (A - C)$ ......(1)
Similarly we can show that,
$(A - B) \cap (A - C) \subseteq A - (B \cap C)$ ......(2)
By (1) and (2) we have $A - (B \cap C) = (A - B) \cap (A - C)$.

g. Prove that the total degree of a graph is always an even number.

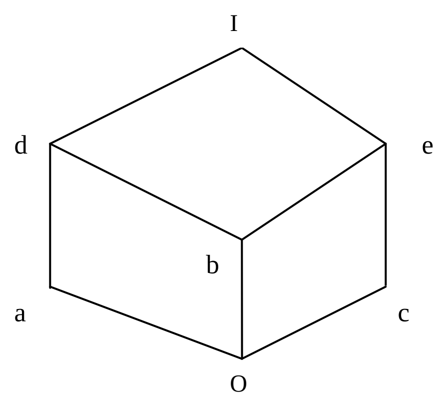
Answer: Let G be a graph with n-vertices and e-edges. Then total degree of graph G is denoted as

D(G) = sum of the degrees of all the vertices = $\Sigma d(v_i)$ for i = 1 to n ….(1)

On the other hand we know that an edge is associated with 2 degrees in a graph. Thus e-edges will contribute (2e) degrees is a graph G. Hence D(G) = 2e …..(2)

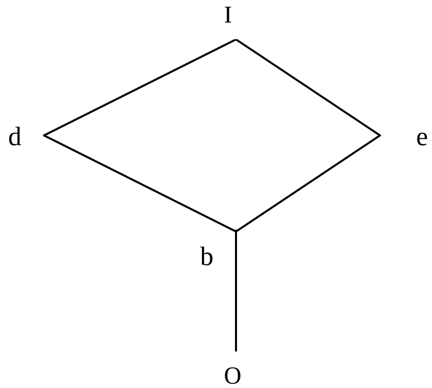
By (1) and (2), we have $D(G) = \Sigma d(v_i) = 2e$ = even number.

Q.2 a. Consider the lattice L given below:



(i) Find all sub-lattices with 5-elements.
(ii) Find atoms.
(iii) Find complement of a and b if they exists.
(iv) Is L distributive? Complemented?

Answer):
(i) sub-lattice with 5 elements is shown below:



It should be noticed that every pair of element in the above diagram have a least upper bound and a greatest lower bound. Hence they are lattice, and since all these elements are from the original lattice, it is a sub lattice.

(i) **atoms:** A non-zero element a is called atom if for al x in lattice L,

$x \wedge a = a$ or $x \wedge a = O$

There are 3 atoms a, b and c in this lattice as

$a \wedge a = a$	$b \wedge a = O$	$c \wedge a = O$
$a \wedge b = O$	$b \wedge b = b$	$c \wedge b = O$
$a \wedge c = O$	$b \wedge c = O$	$c \wedge c = c$
$a \wedge d = a$	$b \wedge d = b$	$c \wedge d = O$
$a \wedge e = O$	$b \wedge e = b$	$c \wedge e = c$
and $a \wedge I = a$	$b \wedge I = b$	$c \wedge I = c$

(ii) Complement of an element a in lattice L is a' if $a \vee a' = I$, $a \wedge a' = O$

There are two complements of a. They are c and e, as

$a \vee c = I$, $a \wedge c = O$ and $a \vee e = I$, $a \wedge e = O$

Complement of b does not exist here.

(iii) Since element 'a' has two complements, lattice is not distributive.

(In a distributive, complemented lattice, complements are always unique)

b. In a Boolean algebra prove that $(a+b)' = a' \cdot b'$

Answer:

To prove that $a' \cdot b'$ is the complement of $(a+b)$, we need to show that

(i) $(a+b)+(a' \cdot b') = 1$

(ii) $(a+b).(a' \cdot b') = 0$

For (i) by distributive law, we have:

$(a+b)+(a' \cdot b') = [(a+b)+a'].[(a+b)+b']$

$= [a+(b+a')].[a+(b+b')]$ (by associativity)

$= [a+(a'+b)].[a+1]$ as $b+b' = 1$ (by commutativity)

$= [(a+a')+b].[a+1]$

$=[1+b].[a+1] = 1.1 = 1$ (as a+1 =1, Dominance law)

Now for (ii), again by distributive law we have:

$(a+b).(a'\cdot b') = [a.(a'.b')]+[b.(a'.b')]$

$= [(a.a').b')]+[(b.b').a']$ (By associativity and commutativity)

$= [0.b']+[0.a']$ (as a.a' = 0, compliment law)

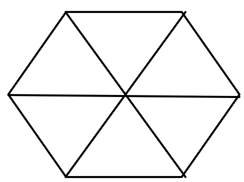
$= 0 + 0 = 0$

Hence $(a+b)' = a'\cdot b'$

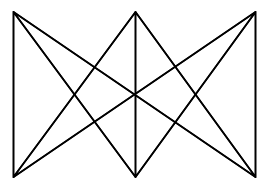
Q.3 a. Prove that $K_{3,3}$ is not a planar graph.

Answer:

$K_{3,3}$ with 6 vertices and 9 edges is drawn as:



OR



We can see that each region in the graph is bounded by 4-edges and since each edge is also associated with 2-regions, we have: 2e = 4r ⇒ e = 2r

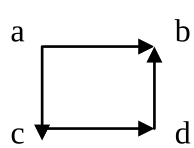
By Euler's formula r = e – n + 2. Also we have e = 2r = 2(e – n + 2)

e = 2e – 2n + 4 => e = 2n – 4

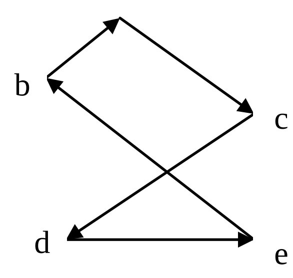
For $K_{3,3}$ e = 2.6 – 4 = 8 (as n = 6 and e = 9)

But we have 9 edges in $K_{3,3}$. So we obtained a contradiction here and hence $K_{3,3}$ in non-planar.

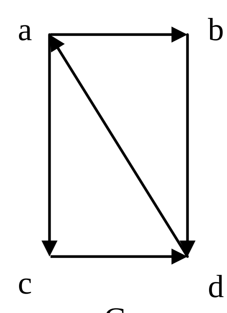
b. Which of the following graphs have an Euler circuit? Identify the one that is not Euler circuit but an Euler path.



G_1



G_2



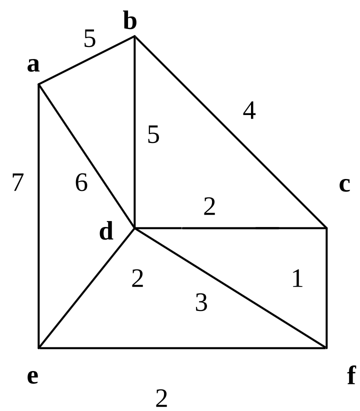
G_3

Answ

Euler circuit in a directed graph:- A directed circuit in a graph is called Euler if it covers all the edges of the graph. A directed walk which covers all the edges of the graph is called Euler line.

Graph G_2 has Euler circuit as (a→c→d→e→b→a)
Graph G_3 has Euler line as (a→b→d→a→c→d)

Q.4 a. Apply Prim's algorithm to determine the minimal spanning tree in the given graph:



Answer:

	a	b	c	d	e	f
a	-	5	∞	6	7	∞
b	5	-	4	5	∞	∞
c	∞	4	-	2	∞	1
d	6	5	2	-	2	3
e	7	∞	∞	2	-	2
f	∞	∞	1	3	2	-

Now according to first step,

(i) Identify the least element in first row and that is 5. Thus edge (a, b) is selected.

(ii) Now search the least element in the rows corresponding to a and b. The next edge will be (b, c) as 4 is the least in both the rows.

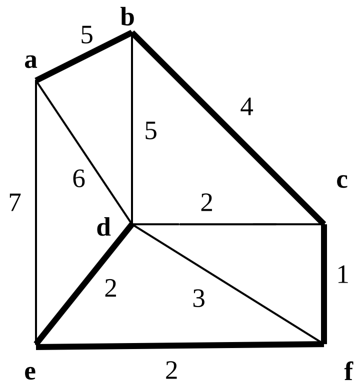
(iii) Again search for the least element in the rows corresponding to a, b and c. The next searched edge is (c, f).

(iv) Similarly we will proceed and hence the next search edges will be (f, e) (e, d).

(v) Since all the 5 edges of the spanning tree have been searched, we will stop the process.

(vi) The total weight of the tree is 5+4+1+2+2 = 14 units.

(vii) The minimal spanning tree is shown below in bold:



b. Prove that the maximum possible height of a binary tree is $\frac{(n-1)}{2}$, where n is the number of vertices.

Answer:

To construct a binary tree with maximum possible height i.e; $maxL_{max}$, we must have exactly two vertices at each level. Thus

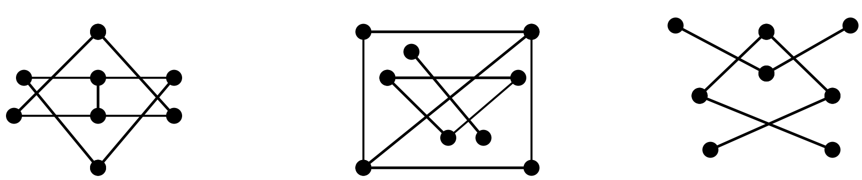
Total number of vertices n = $2^0 + 2^1 + 2^1 + \ldots + 2^1$ (p-level)

$$n = 1 + 2.p \Rightarrow p = \frac{(n-1)}{2}$$

Maximum level of any vertex in a binary tree is called the height of the tree.

$$\text{Thus } maxL_{max} = p = \frac{(n-1)}{2}$$

c. Check whether the following graphs are connected. If not find out the number of components.



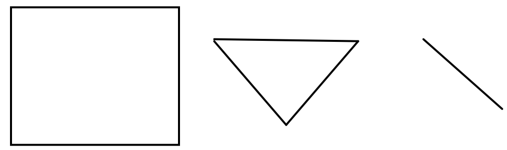
G_1 **G_2** **G_3**

Answer:

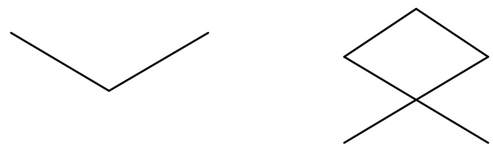
Graph is said to be connected if there exist path between each and every vertex. Thus graph G_1 is connected.

Graph G_2 and G_3 are not connected and have 3 and 2 components respectively.

G_2:



G_3:



Q.5 a. Prove that $1+\frac{1}{4}+\frac{1}{9}+.....+\frac{1}{n^2} \leq 2-\frac{1}{n}\ , \quad \forall n \in N$ **.**

Answer:

Basic step: p(1) is true

$$p(1) = 1 \leq 2 - \frac{1}{1} = 1$$

Induction step: Let us assume that p(k) is true for some k$\in$N.

$$p(k) = 1+\frac{1}{4}+\frac{1}{9}+.....+\frac{1}{k^2} \leq 2-\frac{1}{k}\ , \quad \forall k \in N$$

Now consider p(k+1)

$$p(k+1) = 1+\frac{1}{4}+\frac{1}{9}+.....+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq (2-\frac{1}{k})+\frac{1}{(k+1)^2} \qquad ....(1)$$

Now $(2-\frac{1}{k})+\frac{1}{(k+1)^2}+\frac{1}{k+1}-\frac{1}{k+1}$

$$= 2-\frac{1}{k+1}+\left[\frac{1}{(k+1)^2}+\frac{1}{(k+1)}-\frac{1}{k}\right]$$

$$= 2-\frac{1}{k+1}+\left[\frac{k+k(k+1)-(k+1)^2}{k(k+1)^2}\right]$$

$$= 2-\frac{1}{k+1}-\left[\frac{1}{k(k+1)^2}\right]$$

$$= 2-\frac{1}{k+1}+negative \quad value \leq 2-\frac{1}{k+1}$$

Hence from (1) $p(k+1) = 1+\frac{1}{4}+\frac{1}{9}+.....+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 2-\frac{1}{k+1}$

b. Let A = {1, 2, 3, 4}. Relations R and S are defined on (A$\times$A) as:
R = {(a, b) | a+b ≤ 7, a and b$\in$A} and
S = {(a, b) | a+b ≥ 3, a and b$\in$A}
(i) List all the elements of relation R and S.
(ii) Find R^{-1}, S^{-1} and RoS.

(iii) **Show that $(RoS)^{-1} = S^{-1}oR^{-1}$**

Answer:

(i) R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 3)}
S = {(1, 4), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}

(ii) R^{-1} = {(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (3, 3)}
S^{-1} = {(4, 1), (3, 2), (4, 2), (3, 3), (4, 3), (4, 4)}
RoS = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)}

(i) $(RoS)^{-1}$ = {(3, 1), (4, 1), (3, 2), (3, 3), (4, 2), (4, 3)}
$S^{-1}oR^{-1}$ = {(4, 1), (3, 1), (3, 2), (4, 2), (4, 3), (3, 3)} = $(RoS)^{-1}$

Q.6 a. Show that $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow s)] \rightarrow (s \vee r)$ **is a tautology.**

Answer

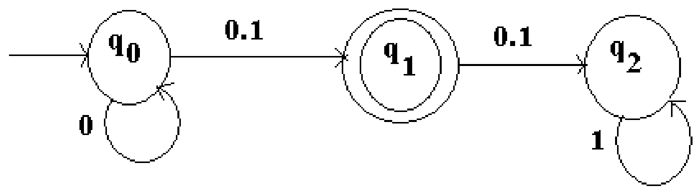
P	Q	R	S	(P∨Q)	(P→R)	(Q→S)	(S∨R)	[(P∨Q)∧(P→R)∧(Q→S)] → (S∨R)
0	0	0	0	0	1	1	0	1
0	0	0	1	0	1	1	1	1
0	0	1	0	0	1	1	1	1
0	0	1	1	0	1	1	1	1
0	1	0	0	1	1	0	0	1
0	1	0	1	1	1	1	1	1
0	1	1	0	1	1	0	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	0	1	1	0	1	1	1
1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	1	1	1
1	1	0	0	1	0	0	0	1
1	1	0	1	1	0	1	1	1
1	1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1	1

b. In a group of athletic teams in a certain institute, 21 are in the basket ball team, 26 in the hockey team, 29 in the foot ball team. If 14 play hockey and basketball, 12 play foot ball and basket ball, 15 play hockey and foot ball, 8 play all the three games.
(i) How many players are there in all?
(ii) How many play only foot ball?

Answer:
Let A, B and C is set of players who play Basket ball, Hockey and Foot ball respectively.
Thus
|A| = 21, |B| =26, |C| = 29,
|A□B| = 14, |A□C| = 12, |B□C| = 15, and
|A□B□C| = 8.

Now for part (i), we have to find $|A \cup B \cup C|$. This can be computed using the formula
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$= 21 + 26 + 29 – 14 – 12 – 15 + 8 = 84 – 41 = 43$.
For part (ii) we have to find number of players playing only football. This is
$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 29 – 12 – 15 + 8 = 10$.

Q.7 Convert the following non-deterministic Automaton into equivalent deterministic machine.



Answer:
The transition table of the given machine M is as follows:

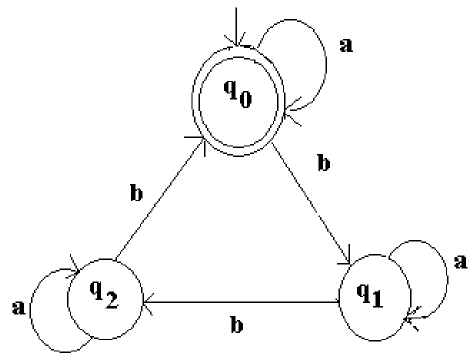
Input→ State↓	0	1
$\rightarrow q_0$	q_0, q_1	q_1
q_1	q_2	q_2
q_2		q_2

Transition function: δ

Also $M = \{Q = (q_0, q_1, q_2), q_0, \Sigma = \{0, 1\}, \{q_2\}, \delta\}$.
The equivalent DFA M_D is as follows:
$Q_D = 2^Q = [\phi, \{q_0\}, \{q_1\}, \{q_2\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}]$
$\Sigma_D = \Sigma = \{0, 1\}$
$q_{0D} = q_0$
$F_D = [\{q_1\}, \{q_0, q_1\},\{q_1, q_2\},\{q_0, q_1, q_2\}]$
Transition function δ_D is given as follows:

Input→ State↓	0	1
$\rightarrow\{q_0\}$	$\{q_0, q_1\}$	$\{q_1\}$
$\{q_1\}$	$\{q_2\}$	$\{q_2\}$
$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$
$\{q_1, q_2\}$	$\{q_2\}$	$\{q_2\}$
$\{q_2\}$	ϕ	$\{q_2\}$

b. Describe the language L(M) accepted by the machine M whose transition graph is given below:



Answer: The transition table for the machine is as follows:

Input→ State↓	a	b
→q_0	q_0	q_1
q_1	q_1	q_2
q_2	q_2	q_0

We can see from the table that for input b the machine follows a path from q_0 |---- q_1 |---- q_2 which is the final state. So on receiving 3 b's in a string the machine will reach the final state. So there is a loop from initial to final state with 3 b's, hence the number of b's in a string should be a multiple of 3 to get accepted by the machine.

Thus L(M) = {all strings over (a, b) where number of b's are multiple of 3}

Text Books

1. J.L. Hein, Discrete Structures, Logic and Computability, Jones and Bartlett Publishers, 3rd Edition

2. R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Pearson Education, 1994