Q.2 a. Evaluate $\underset{x\to 0}{lt}\dfrac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$ **(8)**

Answer:

$$\underset{x\to 0}{lt}\frac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}=\underset{x\to 0}{lt}\frac{\left[1+x+\frac{x^2}{2!}+....\right]\left[x-\frac{x^3}{3!}+....\right]-x-x^2}{x^2+x\left[-x-\frac{x^2}{2}-....\right]}\ 3$$

[1 mark for each correct expansion]

$$=\underset{x\to 0}{lt}\frac{\left[x+x^2+\frac{x^3}{3}....\right]-x-x^2}{x^2-\left[x^2+\frac{x^3}{2}+....\right]}\ 2=\underset{x\to 0}{lt}\frac{\left[\frac{x^3}{3}+....\right]}{\left[-\frac{x^3}{2}-....\right]}\ 2=-\frac{2}{3}\ 1$$

b. Find the area enclosed by the curve $a^2x^2=y^3(2a-y)$. **(8)**

Answer:

The required area $A=2\int_0^{2a}xdy\ 1=\frac{2}{a}\int_0^{2a}y\sqrt{y(2a\_y)}dy\ 3=\pi a^2\ 4$

Q.3 a. Separate $\sin^{-1}(\cos\theta+i\sin\theta)$ into real and imaginary parts. **(8)**

Answer:

Let $\sin^{-1}(\cos\theta+i\sin\theta)=x+iy\ 1\ \therefore\cos\theta+i\sin\theta=\sin(x+iy)$

$\cos\theta+i\sin\theta=\sin x\cosh y+i\cos x\sinh y,\therefore\cos\theta=\sin x\cosh y\ 1,\ \sin\theta=\cos x\sinh y\ 1$

Squaring and adding we get $1=\sin^2x+\sinh^2y\ 1\Rightarrow\cos^2x=\sinh^2y\ 1\Rightarrow\cos^2x=\sin\theta\ 1$

$\therefore x=\cos^{-1}\sqrt{\sin\theta}\ 1$ *and* $y=\log\left[\sqrt{\sin\theta}+\sqrt{(1+\sin\theta)}\right]\ 1$

b. Find the moment about the point M(-1,-2,3) of the force represented in magnitude And position by $\overrightarrow{AB}$ where the point A and B have the coordinates (1,2,-3) and (1,-2,3) respectively. **(8)**

Answer:

$$\vec{F}=\overrightarrow{AB}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)-\left(\hat{i}+2\hat{j}-3\hat{k}\right)=\left(-4\hat{j}+6\hat{k}\right)\ 2$$

$$\vec{r}=\overrightarrow{MA}=\left(\hat{i}+2\hat{j}-3\hat{k}\right)-\left(-\hat{i}-2\hat{j}+3\hat{k}\right)=\left(2\hat{i}+4\hat{j}-6\hat{k}\right)\ 2$$

$$Moment=\vec{r}\times\vec{F}\ 1=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&-4&6\\2&4&-6\end{vmatrix}\ 1=\left(12\hat{j}+8\hat{k}\right)\ 2$$

Q.4 a. A resistance of 100 ohms, an inductance of 0.5 henry are connected in a series with a battery of 20 volts. Find the current in the circuit as a function of time. **(8)**

Answer:

$I.F. = e^{\frac{R}{L}t}$ (2)

By Kirchoff's first law, we have

$$Ri + L\frac{di}{dt} = E \text{ (1) } \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \text{ (1) } \Rightarrow i = \frac{E}{R}\left[1 - e^{\frac{-R}{L}t}\right] \text{ (2)}$$

R=100 ohms, L = 0.5 henry, E = 20 Volts, $i = \frac{1}{5}\left[1 - e^{-200t}\right]$ (1)

(1)

b. Find the Fourier series of the function $f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$ **(8)**

Answer:

Since f(x) is an even function, hence $b_n = 0$, (1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \text{ (1) } = \frac{2}{\pi}\left[\int_0^{\pi}(-x)dx\right] = -\pi \text{ (1)}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \text{ (1) } = \frac{2}{\pi}\int_0^{\pi} -x\cos nx dx = \frac{2}{\pi n^2}\left[1-(-1)^n\right] \text{ (3) } = \begin{cases} 0, & n \text{ is even} \\ \frac{4}{\pi n^2} & n \text{ is odd} \end{cases}$$

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right] \text{ (1)}$$

Q.5 a. Find the Laplace transform of $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$

if $f(t + 2a) = f(t)$ **(8)**

Answer:

The given function is periodic with period 2a.

$$\therefore Lf(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)e^{-st}dt \text{ (2) } = \frac{1}{1-e^{-2as}}\left[\int_0^{a} f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt\right] \text{ (2)}$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right] \text{ (1) } = \frac{1}{s^2}\frac{1-e^{-as}}{1+e^{-as}} \text{ (2)}$$

(1) for correct evaluation of each integral (1) mark

b. Evaluate $L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right)$ **(8)**

Answer:

We know that $L^{-1}\left(\frac{s}{s^2+1}\right)=\cos x$, 1 $L^{-1}\left(\frac{2}{s^2+2^2}\right)=\sin 2x$ 1

statement of convolution theorem 1

$$L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right)=\frac{1}{2}L^{-1}\left(\frac{s}{(s^2+1)}\frac{2}{(s^2+4)}\right)$$

$$=\frac{1}{2}\int_0^t \sin 2x\cos(t-x)dx=\frac{1}{3}(\cos t-\cos 2t)$$ 2 3

Q.6 a. Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e 1.1$ correct to 4 decimal places. **(8)**

Answer:

Let

$f(x)=\log_e x$ $\qquad f(1)=0$

$f'(x)=\frac{1}{x}$, 1 $\qquad f'(1)=1$

$f''(x)=-\frac{1}{x^2}$, 1 $\qquad f''(1)=-1$

$f'''(x)=\frac{2}{x^3}$, 1 $\qquad f'''(1)=2$

$f^{iv}(x)=-\frac{6}{x^4}$, 1 $\qquad f^{iv}(0)=-6$

etc. $\qquad$ etc.

Substituting these values in the Taylor's series

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\ldots,$$ 1

we get

$$\log x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots$$ 1

Now putting $x=1.1$, so that $x-1=0.1$, we have

$$\log(1.1)=1.1-\tfrac{1}{2}(0.1)^2+\tfrac{1}{3}(0.1)^3-\tfrac{1}{4}(0.1)^4+\ldots$$
$$=0.1-0.005+0.0003-0.00002+\ldots=0.0953.$$ 2

b. Using Laplace transforms evaluate the integral $\int_0^\infty \frac{\sin mt}{t}\,dt$ if $m > 0$ **(8)**

Answer:

$$L(\sin mt) = \frac{m}{s^2+m^2} = f(s) \quad L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{m}{s^2+m^2}\,ds = \tan^{-1}\left(\frac{s}{m}\right)_s^\infty$$

1 2 1 2

$$\int_0^\infty e^{-st}\frac{\sin mt}{t}\,dt = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{m}\right)$$ taking limit as s tends to 0, we get

2

$$\int_0^\infty \frac{\sin mt}{t}\,dt = \frac{\pi}{2} \text{ if } m>0, \cancel{-\frac{\pi}{2} \text{ if } m<0}$$

2

Q.7 a. Solve by Laplace transform $\frac{d^2x}{dt^2} + 9x = \cos 2t$, if $x(0) = 1, x\left(\frac{\pi}{2}\right) = -1$ **(8)**

Answer:

Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2\bar{x} - s\,x(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2+4}$$ 1 1

$$(s^2+9)\,\bar{x} = s + a + \frac{s}{s^2+4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$ 2

or $$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}$$ 2

On inversion, we get $x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$ 1

When $t = \pi/2$, $-1 = -\frac{a}{3} - \frac{1}{5}$ or $\frac{a}{3} = \frac{4}{5}$ $\left[\because x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$. 1

b. Solve $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=\left(1-e^x\right)^2$ **(8)**

Answer:

Given equation in *symbolic form* is $(D^2+D+1)y=(1-e^x)^2$

(i) To find C.F.

Its A.E. is $D^2+D+1=0$, $\therefore$ $D=\frac{1}{2}(-1+\sqrt{3}i)$ 2

Thus C.F. $=e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$ 1

(ii) To find P.I.

$$\text{P.I.}=\frac{1}{D^2+D+1}(1-2e^x+e^{2x})=\frac{1}{D^2+D+1}(e^{0x}-2e^x+e^{2x}) \quad 1$$

$$=\frac{1}{0^2+0+1}e^{0x}-2\cdot\frac{1}{1^2+1+1}e^x+\frac{1}{2^2+2+1}e^{2x}=1-\frac{2}{3}e^x+\frac{e^{2x}}{7} \quad 1 \quad 1 \quad 1$$

(iii) Hence the C.S. is $y=e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)+1-\frac{2}{3}e^x+\frac{e^{2x}}{7}$. 1

Q.8 a. Given that $f(x)=e^{-x}\ for -l<x<l$ find the Fourier expansion of f(x). **(8)**

Answer:

$$\text{Let } e^{-x}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \quad 1$$

$$a_0=\frac{1}{l}\int_{-l}^{\cancel{\pi}\,l}f(x)dx=\frac{1}{l}\int_{-l}^{l}e^{-x}dx=\frac{2\sinh l}{l} \quad 1$$

$$a_n=\frac{1}{l}\int_{-l}^{l}e^{-x}\cos\frac{n\pi x}{l}dx=\frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2}=0 \quad 2$$

$$b_n=\frac{1}{l}\int_{-l}^{l}e^{-x}\sin\frac{n\pi x}{l}dx=\frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2} \quad 3$$

$$e^{-x}=\frac{\sinh l}{l}+\sum_{n=1}^{\infty}\frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2}\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}\frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}\sin\frac{n\pi x}{l} \quad 1$$

b. Two circuits of impedances 1+ 2j ohms and 2 + 3j ohms are connected in parallel and a.c. voltage of 50 volts is applied across the parallel combination. Calculate the magnitude of the current as well as power factor for each circuit and the magnitude of the total current for the parallel combination and its power factor. **(8)**

Answer:

$z_1 = 1+2j,\ z_2 = 2+3j,\ i_1 = \frac{V}{z_1} = \frac{50}{1+2j} = 10-20j$ (1) $i_2 = \frac{V}{z_2} = \frac{50}{2+3j} = \frac{100}{13} - \frac{150}{13}j$ (1)

Magnitude of $i_1 = \sqrt{100+400} = 10\sqrt{5}ampere$ (1). Magnitude of $i_2 = \frac{50}{13}\sqrt{13}ampere$ (1)

Power factor $= \frac{R_1}{|z_1|} = \frac{1}{\sqrt{5}}$ (1) Power factor $\frac{R_2}{|z_2|} = \frac{2}{\sqrt{13}}$ (1)

$i = i_1 + i_2 = \frac{230-410j}{13}$ (1) Magnitude $= \frac{10}{13}\sqrt{529+1681} = \frac{110}{13}\sqrt{10}$ (1)

Q.9 a. IF $\left|\vec{A}+\vec{B}\right| = 50, \left|\vec{A}-\vec{B}\right| = 30, \left|\vec{B}\right| = 10,$ find $\left|\vec{A}\right|$ **(8)**

Answer:

$\left|\vec{A}+\vec{B}\right| = 50, \left|\vec{A}+\vec{B}\right|^2 = 2500 \Rightarrow \left(\vec{A}+\vec{B}\right).\left(\vec{A}+\vec{B}\right) = 2500 \Rightarrow \left|\vec{A}\right|^2 + \left|\vec{B}\right|^2 + 2\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta = 2500$ (2) (1)

$\left|\vec{A}-\vec{B}\right| = 30, \left|\vec{A}-\vec{B}\right|^2 = 900 \Rightarrow \left(\vec{A}-\vec{B}\right).\left(\vec{A}-\vec{B}\right) = 900 \Rightarrow \left|\vec{A}\right|^2 + \left|\vec{B}\right|^2 - 2\left|\vec{A}\right|\left|\vec{B}\right|\cos\theta = 900$ (2) (1)

Adding we get, $\left|\vec{A}\right|^2 + \left|\vec{B}\right|^2 = 1700 \Rightarrow \left|\vec{A}\right|^2 + (10)^2 = 1700 \Rightarrow \left|\vec{A}\right| = 40$ (1) (1)

b. Evaluate $\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}dx$ then find the value of $\int_0^1 x^n \sin^{-1} x dx$ **(8)**

Answer:

$$\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}dx = a^n \int_0^{\pi/2} \sin^n \theta d\theta,\ x = a\sin\theta$$ (1) integrating we get (1)

$$= \begin{cases} \frac{(n-1)(n-3)....2}{n(n-2)....3}, & n = odd \\ \frac{(n-1)(n-3)....1}{n(n-2)....2}\frac{\pi}{2}, & n = even \end{cases}$$ (1) (1)

$$\int_0^1 x^n \sin^{-1} x dx = \left(\sin^{-1} x \frac{x^{n+1}}{n+1}\right)_0^1 - \int_0^1 \frac{x^{n+1}}{(n+1)\sqrt{1-x^2}}dx$$ (2)

$$= \frac{1}{n+1}\begin{cases} \frac{\pi}{2} - \frac{n(n-2)(n-4)....2}{(n+1)(n-1)(n-3)....3}, & n = even \\ \frac{\pi}{2} - \frac{n(n-2)(n-4)....1}{(n+1)(n-1)(n-3)....2}\frac{\pi}{2}, & n = odd \end{cases}$$ (1) (1)

TEXT BOOKS

1. Engineering Mathematics –Dr. B.S.Grewal, 12th edition 2007, Khanna publishers, Delhi
2. Engineering Mathematics – H.K.Dass, S. Chand and Company Ltd, 13th Revised Edition 2007, New Delhi
3. A Text book of Engineering Mathematics – N.P. Bali and Manish Goyal , 7th Edition 2007, Laxmi Publication(P) Ltd