Solution

Q.2 a. Differentiate the following functions: (8)

(i) $x^n\ e^x\ \log_e x$ **(ii)** $\operatorname{cosec}^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)+\cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

Answer: (i)

Let $y = x^n \cdot e^x \cdot \log_a x$

$= x^n \cdot e^x \cdot \dfrac{\log x}{\log a}$

Differentiating both sides w.r.to 'x', then

$\dfrac{dy}{dx} = \dfrac{1}{\log a}\dfrac{d}{dx}\left[x^n \cdot e^x \cdot \log x\right]$

$= \dfrac{1}{\log a}\left[x^n \log x \dfrac{d}{dx}(e^x) + x^n e^x \dfrac{d}{dx}(\log x) + e^x \log x \dfrac{d}{dx}(x^n)\right]$

$= \dfrac{1}{\log a}\left[x^n \log x \cdot e^x + \dfrac{x^n e^x}{x} + e^x \log x \cdot n x^{n-1}\right]$

$= \dfrac{1}{\log a} \cdot e^x x^{n-1}\left[x \log x + 1 + n \log x\right]$

Hence, $\dfrac{dy}{dx} = \dfrac{e^x x^{n-1}}{\log a}\left[x \log x + n \log x + 1\right]$

(ii)

Let $y = \operatorname{cosec}^{-1}\left\{\dfrac{x^2+1}{x^2-1}\right\} + \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

$= \sin^{-1}\left(\dfrac{x^2-1}{x^2+1}\right) + \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

$\because \operatorname{cosec}^{-1} x = \sin^{-1}\left(\dfrac{1}{x}\right)$

$= \dfrac{\pi}{2}$ $\quad \because \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

or $y = \dfrac{\pi}{2} \Rightarrow \dfrac{dy}{dx} = 0.$

b. If $y = \sin(m\sin^{-1} x)$, **then prove that** $(1-x^2)y_{n+1} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$. **(8)**

Answer:

let $y = \sin(m \sin^{-1} x)$ ———— ①

$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$

cross multiplying

$\sqrt{1-x^2}\, y_1 = m \cos(m \sin^{-1} x)$

Squaring on both sides

$(1-x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x)$

$= m^2 \sqrt{1 - \sin^2(m \sin^{-1} x)}$

$(1-x^2) y_1^2 = m^2 (1-y^2)$

Again, diff. w.r.t. to x

$(1-x^2)\, 2 y_1 y_2 - 2x y_1^2 = m^2(-2y y_1)$

$2y_1 [(1-x^2) y_2 - x y_1 + m^2 y] = 0$

$$2y_1 \neq 0$$

$$(1-x^2)y_2 - xy_1 + m^2 y = 0$$

Applying Leibnitz's theorem for n times differentiation.

$$\left[(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n\right] - \left[xy_{n+1} + {}^nC_1(1)y_n\right] + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Q.3 Evaluate

a. $\int \frac{\sin x}{\sin(x-a)}\,dx$ **(8)**

Answer:

$$\int \frac{\sin x}{\sin(x-a)}\,dx = \int \frac{\sin[(x-a)+a]}{\sin(x-a)}\,dx$$

$$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}\,dx$$

$$= \int \cos a\,dx + \sin a \int \frac{\cos(x-a)}{\sin(x-a)}\,dx$$

$$= x\cos a + \sin a \cdot \log\sin(x-a) + c$$

$$\therefore \int \frac{\sin x}{\sin(x-a)}\,dx = x\cos a + \sin a \log\sin(x-a) + c$$

b. $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$ **(8)**

Answer:

Let $I = \int_0^{\frac{\pi}{2}} \cos^7 x\, dx$

$= \int_0^{\frac{\pi}{2}} \cos^6 x \cdot \cos x\, dx$

$= \int_0^{\frac{\pi}{2}} (\cos^2 x)^3 \cdot \cos x\, dx$

$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)^3 \cos x\, dx$

Put $\sin x = t$, $\cos x\, dx = dt$

when $x = 0$, $t = 0$, when $x = \pi/2$ $t = 1$.

$\therefore I = \int_0^1 (1 - t^2)^3\, dt$

$= \int_0^1 \{1 - t^6 - 3t^2(1 - t^2)\}\, dt$

$= \int_0^1 (1 - t^6 - 3t^2 + 3t^4)\, dt$

$= \left[t - \frac{t^7}{7} - \frac{3t^3}{3} + \frac{3t^5}{5}\right]_0^1$

$= 1 - \frac{1}{7} - 1 + \frac{3}{5}$

$= \frac{21 - 5}{35} = \frac{16}{35}$

$\therefore \int_0^{\frac{\pi}{2}} \cos^7 x\, dx = \frac{16}{35}$

Q.4 a. Prove that the inverse of a square matrix, if it exists, is unique. (8)

Answer:

Let A be a non-singular matrix, then A is invertible. Let B and C be two inverse matrix of A, then by definition of inverse

$$AB = BA = I \quad \text{---(i)}$$

and

$$AC = CA = I \quad \text{---(ii)}$$

Now $B = BI$

$= B(AC)$ from (ii)

$= (BA)C$ ($\because$ Matrix multiplication is associative)

$= IC$ using (i)

$= C$

Hence, the inverse of non-singular matrix is unique.

b. Find the adjoint of the matrix A, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} \qquad \textbf{(8)}$$

Answer:

Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix}$

Here $A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3$, $A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12$, $A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$

$A_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1$, $A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5$, $A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$

$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11$, $A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$ $A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$

$$\therefore \ \text{adj } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

Hence, $\text{adj } A = \begin{vmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{vmatrix}$

Q.5 a. How many arrangements can be made with the letters of the word MATHEMATICS? In how many of them vowels are together? (8)

Answer:

Here, 11 letters are in the word MATHEMATICS in which there are two A's and two M's and two are T' and all other are distinct, so

Required number of arrangements are

$$\frac{11!}{2! \times 2! \times 2!} = 4989600$$

there are 4 vowels i.e. A, E, A, I.
Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels) out of which M occurs twice, T occurs twice and rest all different. these 8 letters can be arranged in $\frac{8!}{2! \times 2!}$ ways

But the four vowels (A, E, A, I) can be put together in $\frac{4!}{2!}$ ways

Hence total no. of arrangements in which vowels are always together

$$\frac{8!}{2! \times 2!} \times \frac{4!}{2!} = 10080 \times 12$$
$$= 120960$$

b. Find the middle terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^7$. **(8)**

Answer:

Here the expression is

$\left(3x - \frac{x^3}{6}\right)^7$, i.e. $n = 7$, which is an odd number

so $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e. 4th and 5th term are two middle terms

Now $T_4 = T_{3+1} = {}^7C_3 (3x)^{7-3} \left(-\frac{x^3}{6}\right)^3$

$$= (-1)^3\, {}^7C_3 (3x)^4 \left(\frac{x^3}{6}\right)^3$$

$$= -35 \times 81x^4 \times \frac{x^9}{216}$$

$$= -\frac{105x^{13}}{8}$$

and $T_5 = T_{4+1} = {}^7C_4 (3x)^{7-4} \left(-\frac{x^3}{6}\right)^4$

$$= {}^7C_4 (3x)^3 \left(-\frac{x^3}{6}\right)^4$$

$$= 35 \times 27x^3 \times \frac{x^{12}}{1296}$$

Hence, the middle terms are

$$-\frac{105}{8}x^{13} \text{ and } \frac{35x^{15}}{48}$$

Q.6 a. Find the equation of the ellipse, whose foci and eccentricity are $(\pm 2,0)$ *and* $\frac{1}{2}$. (8)

Answer:

Let the equation of the ellipse be

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the coordinates of

foci are $(\pm ae, 0)$

$\therefore \quad ae = 2 \Rightarrow a \times \frac{1}{2} = 2 \qquad (\because e = \frac{1}{2})$

$\Rightarrow a = 4$

Now $\quad b^2 = a^2(1-e^2)$

$= 16(1-\frac{1}{4}) = 12$

Hence, the equation of the ellipse is

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

b. Find the equation to the circle, which passes through the point (-2, 4) and through the points in which the circle $x^2+y^2-2x-6y+6=0$ is cut by the line $3x+2y-5=0$. (8)

Answer:

Let $L \equiv 3x+2y-5=0$ be the equation of line

and $S \equiv x^2+y^2-2x-6y+6=0$ be the equation of circle. Therefore the equation of the required circle is

$S+\lambda L=0$, i.e.

$$x^2+y^2-2x-6y+6+\lambda(3x+2y-5)=0 \qquad \text{(i)}$$

This passes through the point $(-2,4)$, therefore,

$$4+16+4-24+6+\lambda(-6+8-5)=0$$

$$6-3\lambda=0 \Rightarrow \lambda=2$$

On putting $\lambda=2$ in (i), we get

$$x^2+y^2-2x-6y+6+2(3x+2y-5)=0$$

$$\Rightarrow x^2+y^2+4x-2y-4=0$$

which is the equation of the required circle.

Q.7 a. Find the equation of the line which passes through the point (3,4) and the sum of its intercepts on the axes is 14. (8)

Answer:

Let the equation of the line are

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{——— (i)}$$

Equation (i) passes through (3, 4), therefore

$$\frac{3}{a}+\frac{4}{b}=1 \quad \text{——— (ii)}$$

It is given that $a+b=14 \Rightarrow b=14-a$

putting the values of b in (ii), we get

$$\frac{3}{a}+\frac{4}{14-a}=1 \Rightarrow a^2-13a+42=0$$

$$\Rightarrow (a-7)(a-6)=0$$

$$a=6,\ 7$$

when $a=6$, $b=8$

when $a=7$ $b=7$. $\Rightarrow$ On putting in (i), we get

$$\frac{x}{7}+\frac{y}{7}=1 \text{ and } \frac{x}{6}+\frac{y}{8}=1$$

$\Rightarrow$ $x+y=7$; $4x+3y=24$. Ans.

b. **Find the area of a triangle whose vertices are A (3,2), B (11,8) and C (8,12). (8)**

Answer:

Let $A=(x_1,y_1)=(3,2)$

$B=(x_2,y_2)=(11,8)$

and $C=(x_3,y_3)=(8,12)$, then

Area of Δ ABC $=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

$$=\frac{1}{2}\left[3(8-12)+11(12-2)+8(2-8)\right]$$

$$=\frac{1}{2}\left[3\times -4+11\times 10+8\times -6\right]$$

$$=\frac{1}{2}\left[-12+110-48\right]$$

$$=\frac{1}{2}\left[110-60\right]$$

$$\frac{1}{2}\left[50\right]=25 \text{ Ans.}$$

Q.8 a. Find the differential equation of all circles in xy plane. (8)

Answer:

We know that the general equation of any circle in the x-y plane is

$$x^2+y^2+2gx+2fy+c=0 \quad \text{—— (i)}$$

where g, f and c are arbitrary constant.

Diff. (i) w.r.to 'x' for three times, then

$$2x+2yy'+2g+2fy'=0$$

$$\text{or } x+yy'+g+fy'=0 \quad \text{—— (ii)}$$

Again,

$$1+yy''+y'^2+fy''=0 \quad \text{—— (iii)}$$

and

$$yy'''+y'y''+2y'y''+fy'''=0$$

$$\text{or } yy'''+3y'y''+fy'''=0 \quad \text{—— (iv)}$$

On eliminating f b/w (3) and (4). For the same multiply (3) by y''' and (4) by y'' and substract (4) from (3), we get

$$y'''+yy''y'''+y'^2y'''-yy''y'''-3y''^2y'=0$$

$$y'''(1+y'^2)-3y'y''^2=0$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\cdot\left(\frac{d^2y}{dx^2}\right)^2=0$$

which is the required differential equation

b. Solve the differential equation $\dfrac{dy}{dx}+\dfrac{1+\cos 2y}{1-\cos 2x}=0.$ (8)

Q.9 a. Prove that $\cos 80^0 \cos 60^0 \cos 40^0 \cos 20^0 = \frac{1}{16}$ **(8)**

Answer:

Here $\cos 80^\circ \cos 60^\circ \cos 40^\circ \cos 20^\circ = \frac{1}{16}$

L.H.S $\cos 80^\circ \cos 60^\circ \cos 40^\circ \cos 20^\circ$

$= \frac{1}{2}\left[\cos 20^\circ \cos 40^\circ \cos 80^\circ\right]$

$= \frac{1}{2}\left[\cos A \cos 2A \cos 2^2 A\right]$

where $A = 20^\circ$

$= \frac{1}{2}\left[\frac{\sin 2^3 A}{2^3 \sin A}\right]$

$= \frac{1}{16}\left[\frac{\sin 8A}{\sin A}\right] = \frac{1}{16}\left(\frac{\sin 160^\circ}{\sin 20^\circ}\right)$

$= \frac{1}{16}\,\frac{\sin(180^\circ - 20^\circ)}{\sin 20^\circ}$

$= \frac{1}{16}\,\frac{\sin 20^\circ}{\sin 20^\circ} = \frac{1}{16}$ R.H.S

b. If $A + B + C = \pi$ **, Prove that** $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ **(8)**

Answer:

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Here $\angle A + \angle B + \angle C = \pi$

L.H.S $\sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \cos 2B) + \sin 2C$

$= 2 \sin \frac{(2A+2B)}{2} \cos\left(\frac{2A-2B}{2}\right) + \sin 2C$

$= 2 \sin (A+B) \cos(A-B) + \sin 2C$

$= 2 \sin(\pi - C) \cos(A-B) + \sin 2C$

$\because A + B = \pi - C$

$= 2 \sin C \cos(A-B) + 2 \sin C \cos C$

$= 2 \sin C \left[\cos(A-B) - \cos\{\pi - (A+B)\} \right]$

$= 2 \sin C \left[\cos(A-B) - \cos(A+B) \right]$

$= 2 \sin C \left[(\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B) \right]$

$= 2 \sin C \left[\cos A \cos B + \sin A \sin B - \cos A \cos B + \sin A \sin B \right]$

$= 2 \sin C \cdot 2 \sin A \sin B$

$= 4 \sin A \sin B \sin C$

TEXT BOOKS

1. **Applied Mathematics for Polytechnics, H.K. Dass, 8th Edition, CBS Publishers & Distributors.**

2. **A Text book of Comprehensive Mathematics Class XI, Parmanand Gupta , Laxmi Publications (P) Ltd, New Delhi.**

3. **Engineering Mathematics, HK Dass, S Chand and Company Ltd, 13th Edition, New Delhi**