Q.2 a. Show that every analytic function w = f(z) define two families of curves, which from an orthogonal system. (8)

Answer:

Let $w = u(x,y) + iv(x,y) = f(z)$ be analytic fn.
Consider two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$
Diff. w.r.t. x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

∴ Slope of both curves are

$$m_1 = \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \; ; \quad m_2 = \frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$\therefore m_1 m_2 = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = -1 \quad \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ fn being analytic}\right)$$

Product of slopes of curves being -1, curves form an orthogonal system.

b. Show that the transformation $\omega = \sin z$, maps the families of lines x = constant and y = constant into two families of confocal central conics. (8)

Answer:

$w = \sin z = \sin(x+iy)$. Separating into real & imaginary parts

$w = u + iv = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y \; , \; v = \cos x \sinh y$

eliminating y, $$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$ which shows that lines

x = constant (lines parallel to y axis) in z plane maps into hyperbolas in w plane.

eliminating x, $$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$ Which shows that lines

y = constant (lines parallel to x-axis) in z plane maps into ellipses in w-plane. Hence the transformation $w = \sin z$ maps lines x = const & y = const into families of confocal central conics

Q.3 a. If $f(\alpha) = \oint_C \frac{3z^2 + 7z + 1}{z - \alpha} dz$, where C is the circle $x^2 + y^2 = 4$, find the values of

(i) F' (1-i) (ii) F''(1-i) **(8)**

Answer:

Let $f(z) = 3z^2 + 7z + 1$

∴ By Cauchy's integral formula

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha} dz \quad \text{where } z=\alpha \text{ is a point within } C$$

$$F(\alpha) = \int_C \frac{f(z)}{z-\alpha} dz = 2\pi i f(\alpha) \quad \text{or} \int \frac{3z^2+7z+1}{z-\alpha} = 2\pi i\left[3\alpha^2+7\alpha+1\right]$$

where $z=\alpha$ is within C

Here $\alpha = 1-i$ which is within $C: x^2+y^2=4$

Hence $F(\alpha) = 2\pi i(3\alpha^2+7\alpha+1)$ at $\alpha = 1-i$

or $F'(\alpha) = 2\pi i(6\alpha+7)$ or $F'(1-i) = 2\pi i\left[6-6i+7\right] = 2\pi(6+13i)$

and $F''(\alpha) = 12\pi i$ or $F''(1-i) = 12\pi i$

b. Find the Laurent's series expression of $\frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$. **(8)**

Answer:

By partial fraction

$$\frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

$$= \frac{1}{z+2-3} - \frac{1}{z+2-5} + \frac{1}{z+2}$$

For the region $3 < |z+2| < 5$,

$$= \frac{1}{z+2}\left[1-\frac{3}{z+2}\right]^{-1} + \frac{1}{5}\left[1-\frac{z+2}{5}\right]^{-1} + (z+2)^{-1}$$

$$= \frac{1}{z+2}\left[1+\frac{3}{(z+2)}+\frac{3^2}{(z+2)^2}+\frac{3^3}{(z+2)^3}+\cdots\right]+\frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{5^2}+\frac{(z+2)^3}{5^3}+\cdots\right]$$

$$= \frac{2}{z+2}+\frac{3}{(z+2)^2}+\frac{3^2}{(z+2)^3}+\cdots+\frac{1}{5}\left[1+\frac{z+2}{5}+\left(\frac{z+2}{5}\right)^2+\left(\frac{z+2}{5}\right)^3+\cdots\right], \quad + \frac{1}{z+2}$$

is read Laurent's series

Q.4 a. Show that Curl Curl $\vec{F}$ = grad div $\vec{F}$ - $\nabla^2\vec{F}$. **(8)**

Answer:

$$\text{Curl curl } \vec{F} = \nabla\times(\nabla\times\vec{F}) = \left(\sum i\frac{\partial}{\partial x}\right)\times\left[\sum\left(i\frac{\partial}{\partial x}\right)\times\vec{F}\right]$$

$$= \left(\sum i\frac{\partial}{\partial x}\right)\times\left[i\times\frac{\partial\vec{F}}{\partial x} + j\times\frac{\partial\vec{F}}{\partial y} + k\times\frac{\partial\vec{F}}{\partial z}\right]$$

$$= \sum i\times\left[i\times\frac{\partial^2\vec{F}}{\partial x^2} + j\times\frac{\partial^2\vec{F}}{\partial x\partial y} + k\times\frac{\partial^2\vec{F}}{\partial x\partial z}\right]$$

$$= \sum\left\{\left[\left(i\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right)i - (i\cdot i)\frac{\partial^2\vec{F}}{\partial x^2}\right] + \left[\left(i\cdot\frac{\partial^2\vec{F}}{\partial x\partial y}\right)j - (i\cdot j)\frac{\partial^2\vec{F}}{\partial x\partial y}\right] + \left[\left(i\cdot\frac{\partial^2\vec{F}}{\partial x\partial z}\right)k - (i\cdot k)\frac{\partial^2\vec{F}}{\partial x\partial z}\right]\right\}$$

$$= \sum\left\{\left(i\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right)i + \left(i\cdot\frac{\partial^2\vec{F}}{\partial x\partial y}\right)j + \left(i\cdot\frac{\partial^2\vec{F}}{\partial x\partial z}\right)k\right\} - \sum\frac{\partial^2\vec{F}}{\partial x^2}$$

$$= \sum i\frac{\partial}{\partial x}\left(i\cdot\frac{\partial\vec{F}}{\partial x} + j\cdot\frac{\partial\vec{F}}{\partial y} + k\cdot\frac{\partial\vec{F}}{\partial z}\right) - \nabla^2\vec{F}$$

$$= \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F} = \text{grad}(\text{div}\vec{F}) - \nabla^2\vec{F} = RHS$$

b. Find the values of λ and μ so that the surface $\lambda x^2y + \mu z^3 = 4$, may cut the surface $5x^2 = 2yz + 9x$ orthogonally at (1, -1, 2). **(8)**

Answer:

(b) Let $f_1 = \lambda x^2y + \mu z^3 - 4 = 0$ and $f_2 = 5x^2 - 2yz - 9x = 0$

$\therefore$ $\nabla f_1 = 2\lambda xy\, i + \lambda x^2 j + \mu 3z^2 k$ and $\nabla f_2 = (10x-9)i - 2zj - 2yk$

At point (1,-1,2)

$\nabla f_1 = -2\lambda i + \lambda j + 12\mu k$ and $\nabla f_2 = i - 4j + 2k$

If two surfaces cut orthogonally, $\nabla f_1\cdot\nabla f_2 = 0$

i.e. $-2\lambda - 4\lambda + 24\mu = -6\lambda + 24\mu = 0$ — (1)

Since point (1,-1,2) lies on both surfaces $\therefore -\lambda + 8\mu - 4 = 0$ — (2)

Solving (1) and (2), we get

$\mu = 1,\ \lambda = 4$

Q.5 a. If $\vec{F} = (5xy - 6x^2)\,i + (2y - 4x)\,j$, evaluate $\int_C \vec{F}.d\vec{R}$ along the curve C in the xy – plane, $y = x^3$ from the point (1, 1) to (2, 8). (8)

Answer:

Here $\vec{F} = (5xy - 6x^2)i + (2y-4x)j$

$\therefore \int_C \vec{F}.d\vec{R} = \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy$

Putting $y = x^3$, where x goes from 1 to 2,

$$= \int_1^2 \left[(5x.x^3 - 6x^2)dx + (2.x^3 - 4x).3x^2 dx\right]$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2 = 35$$

b. Use Divergence theorem to evaluate $\iint_S \vec{F}\cdot\overrightarrow{dS}$, where $\vec{F} = x^3 i + y^3 j + z^3 k$, and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (8)

Answer:

Using divergence theorem

$$\iint_S \vec{F}.d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

Where V is volume bounded by the surface S of sphere $x^2+y^2+z^2 = a^2$

Let us change it to spherical polar coordinates by putting

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

So that $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$ and r changes from 0 to a, θ from 0 to π and ϕ from 0 to 2π.

$$\therefore \iint_S \vec{F}.d\vec{S} = \iiint_V 3r^2.r^2\sin\theta\,dr\,d\theta\,d\phi = 3\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^a r^4\,dr = \frac{12}{5}\pi a^5$$

Q.6 a. Estimate the values of $f(22)$ and $f(42)$ from the following available data: (8)

x:	20	25	30	35	40	45
f(x):	354	332	291	260	231	204

Answer:

Let $x_0 = 20$, $x = 22$, $h = 5$,

$\therefore p = \frac{x - x_0}{h} = \frac{22-20}{5} = 0.4$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

By Newton's forward difference interpolation formula

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{\underline{|2}}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 f(x_0) + \cdots$$

$$f(22) = 354 + (0.4)(-22) + \frac{(.4)(.4-1)}{\underline{|2}}(-19) + \frac{(.4)(.4-1)(.4-2)}{\underline{|3}}\,29 + \frac{(.4)(.4-1)(.4-2)(.4-3)}{\underline{|4}}(-37) + \frac{.4(.4-1)(.4-2)(.4-3)(.4-4)}{\underline{|5}}(45)$$

$$= 354 - 8.8 + 2.28 + 1.856 + 1.5392 + 1.34784 = 352.22304$$

To calculate $f(42)$, we use Newton's Backward difference interpolation formula

Let $x_n = 45$, $x = 42$, $h = 5$, $\therefore p = \frac{x - x_n}{h} = \frac{42-45}{5} = -0.6$

$$f(x) = f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{\underline{|2}}\nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{\underline{|3}}\nabla^3 f(x_n) + \cdots$$

$$\therefore f(42) = 204 + (-.6)(-27) + \frac{(-0.6)(-0.6+1)}{\underline{|2}}(2) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-.6+3)}{\underline{|4}}(8) + \cdots$$

$$= 204 + 16.2 - 0.24 - 0.2688 - 1.02816 = 218.66304$$

b. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's $\frac{1}{3}$ rule, $\int_0^5 \frac{dx}{4x+5}$, by dividing the range into 10 equal parts. (8)

Answer:

Here $y = f(x) = \frac{1}{4x+5}$

Divide range $(0, 5)$ in 10 parts each of width $h = \frac{1}{2}$. The values of $y = f(x) = \frac{1}{4x+5}$ are calculated as shown in the table for various values of x.

x:	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
$f(x)$:	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{17}$	$\frac{1}{19}$	$\frac{1}{21}$	$\frac{1}{23}$	$\frac{1}{25}$
y:	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{\frac{1}{2}}{3}\left[\left(\frac{1}{5} + \frac{1}{25}\right) + 4\left(\frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \frac{1}{23}\right) + 2\left(\frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21}\right)\right] = 0.4024$$

Also by simple integration

$$\int_0^5 \frac{dx}{4x+5} = \frac{1}{4}\log(4x+5)\Big|_0^5 = \frac{1}{4}(\log 25 - \log 5) = \frac{1}{4}\log 5$$

Using above values of $\int_0^5 \frac{dx}{4x+5}$, we get

$$\log 5 = 4(0.4024) = 1.6096$$

Q.7 a. Solve the equation $\frac{y-z}{yz}p+\frac{z-x}{zx}q=\frac{x-y}{xy}$ **(8)**

Answer:

The subsidiary equations of the given system are

$$\frac{yz\,dx}{y-z}=\frac{zx\,dy}{z-x}=\frac{xy\,dz}{x-y}$$

Using multipliers 1,1,1 and x, y, z

$$\text{each fraction}=\frac{yz\,dx+zx\,dy+xy\,dz}{0}=\frac{xyz\,dx+zxy\,dy+xyz\,dz}{0}$$

$\therefore$ $yz\,dx+zx\,dy+xy\,dz=0$ and $dx+dy+dz=0$

Dividing by xyz,

or $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$ Integrating or $x+y+z=c_2$

Integrating $\log(xyz)=\text{const}$ i.e. $xyz=c_1$

Hence complete solution is $x+y+z=f(xyz)$

b. Use Charpit's method to solve:
$2zx - px^2 - 2qxy + pq = 0$ **(8)**

Answer:

Let $f(x,y,z,p,q)=2zx-px^2-2qxy+pq=0$ —(1)

Charpit's subsidiary equations are

$$\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}$$

Substituting values

$$\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2-2pq+2qxy}=\frac{dp}{2z-2qy}=\frac{dq}{0}$$

i.e. $dq=0$ or $q=c$ —(2)

Putting $q=c$ in (1), $p=\frac{2x(z-cy)}{x^2-c}$

Substituting for p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{2x(z-cy)}{x^2-c}dx+c\,dy \quad\text{or}\quad \frac{dz-c\,dy}{z-cy}=\frac{2x\,dx}{x^2-c}$$

Integrating $\log(z-cy)=\log(x^2-c)+\text{const.}$

Hence solution is $(z-cy)=k(x^2-c)$

Q.8 a. A student takes his examination in four subjects A, B, C, D. His chances of passing in A are $\frac{4}{5}$, in B $\frac{3}{4}$, in C $\frac{5}{6}$ and in D are $\frac{2}{3}$. To qualify, he must pass in A and at least two other subjects. What is the probability that he qualifies. (8)

Answer:

Chances of ~~...~~ passing in Subjects A, B, C, D are $\frac{4}{5}, \frac{3}{4}, \frac{5}{6}, \frac{2}{3}$ and therefore chances of not passing in A, B, C, D are $\frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \frac{1}{3}$.

To qualify, student has to pass in A and atleast in two other subjects. Hence, he must pass in A and B and C but not in D or in A, B, D but not in C or in A, C, D but not in B or in all A, B, C, D.

Hence reqd prob. is $= \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{3} + \frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{6} + \frac{4}{5}\frac{1}{4}\frac{5}{6}\frac{2}{3} + \frac{4}{5}\frac{3}{4}\frac{5}{6}\cdot\frac{2}{3}$

$= \frac{4}{5}\left[\frac{15+6+10+30}{72}\right] = \frac{4}{5}\cdot\frac{61}{72} = \frac{61}{90}$

b. A ball is transferred from an urn containing two white and three black balls to another urn containing four white and five black balls. A white ball is then taken out from second urn. What is the probability that the transferred ball is white? (8)

Answer:

A ball is transferred from 1st urn to another urn. It may be white or black. Let A_1 and A_2 be these two events of transferring a white and a black ball. Let B be the event of drawing a white ball from second urn

$\therefore P(A_1) = \frac{2}{2+3} = \frac{2}{5}$, $P(A_2) = \frac{3}{2+3} = \frac{3}{5}$

$P(B/A_1) = \frac{5}{5+5} = \frac{1}{2}$, $P(B/A_2) = \frac{4}{4+6} = \frac{4}{10}$

Hence Prob. ~~that~~ transferred ball is white

$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{\sum P(A_i)P(B/A_i)}$

$= \frac{\frac{10}{50}}{\frac{22}{50}} = \frac{5}{11}$

	1st Urn	2nd Urn	total
$P(A_i)$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{5}{5}=1$
$P(B/A_i)$	$\frac{5}{10}$	$\frac{4}{10}$	
$P(A_i)P(B/A_i)$	$\frac{10}{50}$	$\frac{12}{50}$	$\frac{22}{50}$

Q.9 a. The diameter of an electric cable is assumed to be a continuous variable with p.d.f, f(x) = k x (1-x), $0 \le x \le 1$. Find the value of k and also find mean and variance of x. **(8)**

Answer:

Since $f(x)$ is p.d.f, $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$ or $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$

i.e. $\int_0^1 Kx(1-x)\,dx = 1$ or $K\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$ i.e. $K = 6$

Mean $= \bar{x} = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 x\,Kx(1-x)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$

Variance $= \sigma^2 = \int_{-\infty}^{+\infty} (x-\bar{x})^2 f(x)\,dx$

$= \int_0^1 (x-\bar{x})^2 Kx(1-x)\,dx = 6\int_0^1 (x^2 - 2\bar{x}x + \bar{x}^2)\,x(1-x)\,dx$

$= 6\left[\left(\frac{1}{4} - \frac{1}{5}\right) - 2\bar{x}\left(\frac{1}{3} - \frac{1}{4}\right) + \bar{x}^2\left(\frac{1}{2} - \frac{1}{3}\right)\right]$

$= 6\left[\frac{1}{20} - \frac{1}{12} + \frac{1}{4}\cdot\frac{1}{6}\right] = \frac{1}{20}$

b. A Car- hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5, calculate the proportion of days
(i) on which there is no demand
(ii) on which demand is refused ($e^{-1.5} = 0.2231$) **(8)**

Answer:

The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5

$\therefore P(r) = \frac{e^{-m} m^r}{\lfloor r}$ where r is no. of demands

$= \frac{e^{-1.5}(-1.5)^r}{\lfloor r}$

$\therefore$ Prob. that there is no demand $= P(0) = \frac{e^{-1.5}(-1.5)^0}{\lfloor 0} = e^{-1.5} = 0.2231$

(ii) Prob. that a demand is refused

= Prob. for demand of 3 cars + Prob. for demand of 4 cars + ----

$= P(3) + P(4) + P(5) + \ldots$

$= 1 - P(0) - P(1) - P(2)$ $\left[\because \sum_{r}^{\infty} P(r) = \sum \frac{m^r e^{-m}}{\lfloor r} = 1\right]$

$= 1 - e^{-m} \cdot \frac{m^0}{\lfloor 0} - e^{-m} \frac{m^1}{\lfloor 1} - e^{-m} \frac{m^2}{\lfloor 2}$

$= 1 - e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{\lfloor 2}\right)$

$= 1 - (0.2231)(3.625) = 0.1912625$

TEXT BOOKS

I. Higher Engineering Mathematics –Dr. B.S.Grewal, 40th Edition 2007, Khanna Publishers, Delhi

II. A Text book of engineering Mathematics – N.P. Bali and Manish Goyal , 7th Edition 2007, Laxmi Publication(P) Ltd