Q.2 a. Show that the function f(z) = $e^{-z^{-4}} (z \neq 0)$ and f(0) = 0 is not analytic at z = 0 although the Cauchy-Riemann equations are satisfied at that point. (8)

Answer:

We have, $f(z) = e^{-z^{-4}} = e^{-\frac{1}{(x+iy)^4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$

$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2-4ix^3y+4ixy^3)}$ where, $r^2 = x^2+y^2$

$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)} \cdot e^{4ixy\frac{(x^2-y^2)}{r^8}}$

$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\left[\cos\left\{\frac{4xy(x^2-y^2)}{r^8}\right\} + i\sin\left\{\frac{4xy(x^2-y^2)}{r^8}\right\}\right]$ 4

Now, $\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt}\frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}\frac{e^{-x^{-4}}-0}{x} = \underset{x\to 0}{Lt}\frac{1}{x e^{\frac{1}{x^4}}}$

$= \underset{x\to 0}{Lt}\frac{1}{x\left[1+\frac{1}{x^4}+\frac{1}{2x^8}+\cdots\right]}$

$= \underset{x\to 0}{Lt}\frac{1}{x+\frac{1}{x^3}+\frac{1}{2x^7}+\cdots} = \frac{1}{\infty} = 0$

Similarly, $\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt}\frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\frac{e^{-\frac{1}{y^4}}-0}{y} = 0$

and $\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt}\frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt}\frac{0-0}{y} = 0$

and $\frac{\partial v}{\partial y} = \underset{x\to 0}{Lt}\frac{0-0}{y} = 0$ 6

Hence C-R equations are satisfied at the origin.

Now f(z) is not analytic at z = 0. Because f(z) has an infinite discontinuity at the origin.

Now, we have, $f'(0) = \underset{z\to 0}{Lt}\frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt}\frac{1}{z e^{\frac{1}{z^4}}}$

$= \underset{z\to}{Lt}\frac{1}{re^{i\pi/4}} \cdot \frac{1}{\exp[-r^{-4}]}$ Taking $z = re^{i\pi/4}$

$= \underset{r\to 0}{Lt}\frac{1}{re^{i\pi/4}} \cdot \frac{1}{\exp\left(-\frac{1}{r^4}\right)} = \infty$

Hence f'(z) does not exist.

8

b. Find the bilinear transformation which maps the point z = 1, i,-1 into the points w = i, 0, -i . Hence find the image of $|z| < 1$. **(8)**

Answer:

We have the bilinear transformation,

$$w = \frac{az+b}{cz+d} \quad \text{—— (i)}$$

Now putting the values of w and z in (i), we get,

$$i = \frac{a+b}{c+d},$$

$$0 = \frac{ai+b}{ci+d},$$

$$-i = \frac{-a+b}{-c+d} \qquad 4$$

$$\Rightarrow \quad a+b = i(c+d) \quad \text{—— (ii)}$$

$$b + ia = 0 \quad \text{—— (iii)}$$

$$(-a+b) = i(-c+d) \quad \text{—— (iv)}$$

from (iii) $b = -ia$

Now adding (ii) and (iv),

$$2b - 2ic = 0 \Rightarrow c = \frac{b}{i} = -a$$

Now from (ii) and (iv)

$$2a - 2id = 0 \Rightarrow d = \frac{a}{i} = -ia \qquad 6$$

Putting all these values in (i)

$$w = \frac{az - ia}{-az - ia}$$

$$\Rightarrow \quad w = \frac{i-z}{i+z} \quad \text{—— (v)}$$

Which is the required Bilinear transformation.

Now, from (v), we have

$$z = i\left(\frac{1-w}{1+w}\right)$$

$\therefore$ $|z| < 1$ is mapped into the region.

$$\left| i\left(\frac{1-w}{1+w}\right)\right| < 1 \Rightarrow \frac{|i|\,|1-w|}{|1+w|} < 1$$

$$\Rightarrow \quad |1-w| < |1+w|$$

Put, $w = u+iv$

So, $|1-u-iv| < |1+u+iv|$

$$\Rightarrow \quad (1-u^2) + v^2 < (1+u)^2 + v^2$$

$$\Rightarrow \quad u > 0 \qquad 8$$

Q.3 a. For the function f(z) = $\dfrac{4z-1}{z^4-1}+\dfrac{1}{z-1}$, find all Taylor or Laurent series about the centre zero. (8)

Answer:

we have, $f(z) = \dfrac{4z-1}{z^4-1} + \dfrac{1}{z-1}$

Poles are $z^4-1=0$

$\Rightarrow z = 1, -1, \pm i$ 4

$$\therefore f(z) = -\frac{3}{4(1-z)} + \frac{5}{4(1+z)} + \left(-2z+\frac{1}{2}\right)\frac{1}{1+z^2} - \frac{1}{1-z}$$

$$= -\frac{7}{4}(1-z)^{-1} + \frac{5}{4}(1+z)^{-1} + \left(-2z+\frac{1}{2}\right)(1+z^2)^{-1}$$

$$= -\frac{7}{4}[1+z+z^2+z^3+\cdots] + \frac{5}{4}[1-z+z^2-z^3+\cdots] + \left(-2z+\frac{1}{2}\right)[1-z^2+z^4\cdots]$$

6

$$= -\frac{7}{4}+\frac{7}{4}z-\frac{7}{4}z^2+\frac{7}{4}z^3\cdots+\frac{5}{4}-\frac{5}{4}z+\frac{5}{4}z^2-\frac{5}{4}z^3 + \cdots - 2z + 2z^3 - 2z^5 + \cdots + \frac{1}{2} - \frac{z^2}{2} + \frac{z^4}{2} + \cdots$$

$$= -\frac{3}{2}z - z^2 + \frac{5}{2}z^3 + \cdots \quad \text{Ans}$$

8

b. Find the residue of f(z) = $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and f(z) = $e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane. (8)

Answer:

we have, $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

Putting denominator $=0$

$\therefore (z+1)^2(z^2+4) = 0$

$f(z)$ has a double pole at $z=-1$ and simple pole at $z=\pm 2i$

Now residue at at $z=-1$, 4

$$\lim_{z\to -1}\frac{d}{dz}\left\{(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}.$$

$$= \lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Now residue at $z=2i$

$$\lim_{z\to 2i}\left[(z-2i)\frac{(z^2-2z)}{(z+1)(z-2i)(z+2i)}\right] = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

and residue at $z=-2i$

$$\lim_{z\to 2i}\left[(z+2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right] = \frac{7-i}{25}$$ 6

we have $f(z) = e^z \operatorname{cosec}^2 z$

$= \dfrac{e^z}{\sin^2 z}$ has double poles at $z=0, \pm\pi, \pm 2\pi \cdots$ i.e. $z = m\pi$

8 where $m = 0, \pm, \pm, 2 \cdots$

Residue at $z = m\pi$ is $\displaystyle\lim_{z\to m\pi}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\}$

$$\underset{z \to m\pi}{Lt} \frac{e^{z}\left[(z-m\pi)^2 \sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2 \cos z\right]}{\sin^3 z}$$

Putting $z - m\pi = u \Rightarrow z = u + m\pi$ 4

$$\underset{u \to 0}{Lt}\ e^{u+m\pi} \left\{ \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u} \right\}$$

$$= \underset{u \to 0}{Lt}\ e^{m\pi} \left\{ \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u} \right\}$$

$$= e^{m\pi} \underset{u \to 0}{Lt} \left[\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} \cdot \frac{u^3}{\sin^3 u} \right]$$

$$= e^{m\pi} \underset{u \to 0}{Lt} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} = e^{mu} \quad \text{Ans.}$$ 8

$$\left[\because \sin u = u - \frac{u^3}{\lfloor 3} + \frac{4^5}{\lfloor 5} + \cdots \text{ and } \cos u = 1 - \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \cdots \right]$$

Q.4a. **If f(x, y) =** $\frac{1}{2}\log\left(x^2 + y^2\right)$, **show that grad f =** $\dfrac{\bar{r} - \left(k.\bar{r}\right)\hat{k}}{\left\{\bar{r} - \left(\hat{k}.\bar{r}\right)\hat{k}\right\}.\left\{\bar{r} - \left(\hat{k}.\bar{r}\right)\hat{k}\right\}}$ **(8)**

Answer:

We have $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \therefore \bar{r}\,\hat{k} = z$ —— (i)

Now, $f = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{2(x^2+y^2)}\, 2x = \frac{x}{x^2+y^2}$$

Similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial z} = 0$ 4

$$\therefore \text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j} + 0\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j}}{(x^2+y^2)} = \frac{\bar{r} - z\hat{k}}{(x\hat{i}+y\hat{j})\cdot(x\hat{i}+y\hat{j})}$$

$$= \frac{\bar{r} - z\hat{k}}{(\bar{r} - z\hat{k})\cdot(\bar{r} - z\hat{k})}$$ 6

Now replacing z by $\bar{r}\cdot\hat{k}$, we get

$$\text{grad } f = \frac{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}}{\{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\}\cdot\{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\}}.$$ Hence proved 8

b **Find the angle between the surfaces** $x^2 + y^2 + z^2 = 9$ **and** $z = x^2 + y^2 - 3$ **at the point (2, -1,2).** **(8)**

Answer:

Let $f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $N_1 = \nabla f_1$ at $(2,-1,2)$

$= (2xI + 2yJ + 2zK)$ at $(2,-1,2) = 4I - 2J + 4K$

and $N_2 = \nabla f_2$ at $(2,-1,2)$

$= (2xI + 2yJ - K)$ at $(2,-1,2) = 4I - 2J - K$ 4

Since the angle θ between the two surfaces at the point is the angle between their normals at that point and N_1, N_2 are the normals at $(2,-1,2)$ to the given surfaces,

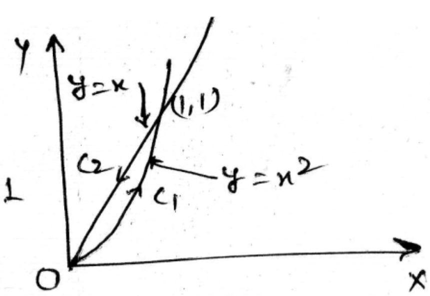
therefore, $\cos\theta = \dfrac{N_1 \cdot N_2}{n_1 \cdot n_2} = \dfrac{(4I-2J+4K)\cdot(4I-2J-K)}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}}$ 6

$= \dfrac{4(4)+(-2)(-2)+4(-1)}{6\sqrt{21}} = \dfrac{16}{6\sqrt{21}}$

Hence the required angle $\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$. Ans 8

Q.5 **a.** **Verify Green's theorem for** $\int_C \left[(xy + y^2)\,dx + x^2 dy\right]$**, where C is bounded by y = x and y =** x^2 **(8)**

Answer:



Here $\phi = xy + y^2$ and $\psi = x^2$

$\therefore \int_C (\phi\,dx + \psi\,dy) = \int_{C_1} + \int_{C_2}$

Along C_1, $y = x^2$ and x varies from 0 to 1

$\therefore \int_{C_1} = \int_0^1 [\{x(x^2) + (x^2)\}^2 dx + x^2 d(x^2)]$

$= \int_0^1 (3x^3 + x^4)\,dx = \dfrac{19}{20}$ 4

Along C_2, $y = x$ and x varies 1 to 0.

$$\therefore \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\}dx + x^2 d(x)] = \int_1^0 3x^2\,dx = -1$$

Thus $\int_C (\phi\,dx + \psi\,dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ —— (i)

Also, $$\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

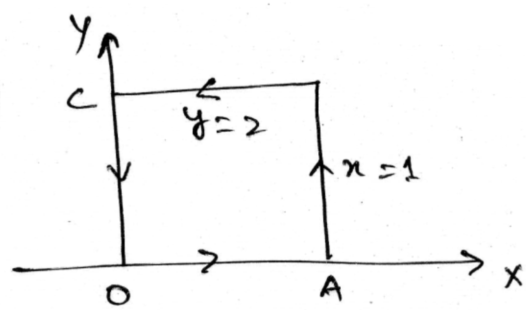
$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\,dy \cdot dx = \int_0^1 [xy - y^2]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = -\frac{1}{20}$$ —— (ii)

Hence Green theorem is verified from the equality of (i) and (ii)

b. Verify Stoke's theorem for $\bar{f} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$ for the surface of a rectangular lamina bounded by x = 0, y = 0, x = 1, y = 2, z = 0. (8)

Answer:



$z = 0$

$$I = \oint_C \bar{F}\cdot d\bar{r} = \oint_C (xy^2\hat{i} + y\hat{j} + z^2x\hat{k}) \cdot (\hat{i}dx \cdot \hat{j}dy)$$

$$= \oint_C xy^2\,dx + y\,dy$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= I_1 + I_2 + I_3 + I_4$$

$I_1 = \int_{OA} xy^2\,dx + y\,dy = 0$, $y = 0$, $x: 0 \to$, $dy = 0$

$I_2 = \int_{AB} xy^2\,dx + y\,dy = \int_0^2 y\,dy = \frac{y^2}{2}\Big|_0^2 = 2$; $x = 1$, $dx = 0$, $y: 0 \to 2$

$I_3 = \int_{BC} xy^2\,dx + y\,dy = \int_1^0 4x\,dx = 2x^2\Big|_0^2 = -2$; $y = 2$, $dy = 0$, $x: 1 \to 0$

$I_4 = \int_{CO} xy^2\,dx + y\,dy = \frac{y^2}{2}\Big|_2^0 = -2$; $x = 0$, $dx = 0$, $y: 2 \to 0$

$I = 2 - 2 - 2 = 0$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y & 0 \end{vmatrix} = -2xy\hat{k}$$

Normal to the surface OABC $= \hat{k}$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot n \, ds = -\iint_S 2xy \, ds$$

$$= -\int_0^2 \int_0^1 2xy \, dx \, dy$$

$$= -\int_0^2 y \, dy = \left. \frac{-y}{2} \right|_0^2$$

$$= -2 \text{ Verified.}$$

8

Q.6 a. The values of a function f(x) are given below for certain values of x:

x	0	1	3	4
f(x)	5	6	50	105

Find the value of f(2) using Langrange's interpolation formula. (8)

Answer:

By Langrange's

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1)$$

$$+ \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

4

$$f(2) = \frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)}(5) + \frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)}(6)$$
$$+ \frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)}(50) + \frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)}(105)$$
$$= \frac{1\times -1\times -2}{-1\times -3\times -4}(5) + \frac{2\times -1\times -2}{1\times -2\times -3}\cdot(6)$$
$$+ \frac{2\times 1 - 2}{3\times 2\times -1}(50) + \frac{2\times 1 - 1}{4\times 3\times 1}(105)$$
$$= -\frac{5}{6} + 4 + \frac{100}{3} - \frac{35}{2} = \frac{-5+24+200-105}{6} = \frac{114}{6} = 19$$

8

$\boxed{\text{Ans: } 19}$

b. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ **using** **(8)**

(i) Simpson's $\frac{1}{3}$ **rule taking h =** $\frac{1}{4}$

(ii) Simpson's $\frac{1}{8}$ **rule taking h =** $\frac{1}{6}$

Hence compute an approximate value of π in each case.

Answer:

(i) The values of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$ are given below,

x :	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	1	0.9412	0.8000	0.6400	0.5000
	y_0	y_1	y_2	y_3	y_4

4

by Simpson's $\frac{1}{3}$ Rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$
$$= \frac{1}{12}\left[(1+0.5000)+4(0.9412+0.6400)+2(0.8000)\right]$$
$$= 0.7854$$

and

Also $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big]_0^1 = \tan^{-1}1 = \frac{\pi}{4}$, $\therefore \frac{\pi}{4} \simeq 0.7854 \Rightarrow \pi \simeq 3.1416$

ans 2

6

(ii) The value of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, 5/6, 1$ are given below,

x :	0	$\frac{1}{6}$	1/3	1/2	2/3	5/6	1
$f(x)$:	1	0.9730	0.9000	0.8000	0.6923	0.5902	0.5000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's Rule $\frac{3}{8}$,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{1}{16}\left[(1+0.5000)+3(0.9730+0.9000+0.6923+0.5902)+2(0.8000)\right]$$

$$= 0.7854$$

8

Also, $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$, $\therefore \frac{\pi}{4} \simeq 0.7854 \Rightarrow \pi \simeq 3.1416$

Q.7 a. Apply Charpit's method to solve $\left(p^2+q^2\right)y = qz$. (8)

Answer:

Here $f = (p^2+q^2)y - qz = 0$ — (1)

$\therefore \frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = p^2+q^2$, $\frac{\partial f}{\partial z} = -q$, $\frac{\partial f}{\partial p} = 2py$, $\frac{\partial f}{\partial q} = 2qy - z$

4

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

or $$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

6

from the first two members, we have $p\,dp + q\,dq = 0$

Integrating, $p^2 + q^2 = a^2$ ——— (ii)

Putting in (i), $q = \dfrac{a^2 y}{z}$

$\therefore$ from (ii), $p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \dfrac{a^4 y^4}{z^2}} = \dfrac{a}{z}\sqrt{z^2 - a^2 y^2}$

$\therefore$ $dz = p\,dx + q\,dy = \dfrac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \dfrac{a^2 y}{z}\,dy$

or $z\,dz - a^2 y\,dy = a\sqrt{z^2 - a^2y^2}\,dx$ or $\dfrac{\frac{1}{2}d(z^2 - a^2y^2)}{\sqrt{z^2 - a^2y^2}} = a\,dx$

Integrating, we get $\sqrt{z^2 - a^2y^2} = ax + b$

or $z^2 = (ax+b)^2 + a^2y^2$ 8

which is the required complete solution.

b. Use the method of separation of variables to solve the equation $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ given that v = 0 when t $\to \infty$, as well as v = 0 at x = 0 at x = 0 and x = 1 **(8)**

Answer:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad \text{——— (i)}$$

Let us assume that $V = XT$ where X is a function of x only and T that of t only.

$$\frac{\partial v}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = T\frac{d^2X}{dx^2}$$ 4

Substituting these values in (i) we get

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

or $$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \quad \text{——— (ii)}$$

Let each side of (ii) be equal to a constant $(-p^2)$

$$\frac{1}{T^2}\frac{dT}{dt} = -p^2 \quad \text{or} \quad \frac{dT}{dt} + p^2T = 0 \quad \text{——— (iii)}$$

and $$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \quad \text{or} \quad \frac{dX^2}{dx^2} + p^2X = 0 \quad \text{——— (iv)}$$

Solving (iii) and (iv) we have

$T = c_1 e^{-p^2 t}$

$X = c_2 \cos px + c_3 \sin px$

$\therefore \quad V = c_1 e^{-p^2 t}(c_2 \cos px + c_3 \sin px)$ ——— (v)

6

Putting $x = 0$, $V = 0$ in (v), we get

$0 = c_1 e^{-p^2 t} c_2$ $\quad \therefore c_2 = 0$, Since $c_1 \neq 0$

On putting the value of c_2 in (v), we get

$V = c_1 e^{-p^2 t} c_3 \sin px$ ——— (vi)

Again putting $x = l$, $V = 0$ in (vi) we get

$0 = c_1 e^{-p^2 t} . c_3 \sin pl$

Since c_3 cannot be zero

$\therefore \quad \sin pl = \sin n\pi$, $\quad \therefore p = \frac{n\pi}{l}$, n is any integer

On putting the value of p in (vi) it becomes

Hence, $V = c_1 c_3 e^{-n^2\pi^2 t/l^2} . \sin \frac{n\pi x}{l}$

$V = b_n e^{\frac{-n^2\pi^2 t}{l^2}} . \sin \frac{n\pi x}{l}$, $\quad b_n = c_1 c_2$

This equation satisfies the given condition for all integral values of n.

Hence taking $n = 1, 2, 3, \ldots$, the most general solution is

$V = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 t}{l}} . \frac{\sin n\pi x}{l}$ Ans.

8

Q.8 a. There are 6 positive and 8 negative numbers. Four numbers are chosen at random, without replacement and multiplied. What is the probability that the product is a positive number? (8)

Answer:

To get from the product of four numbers, a positive number, the possible combinations are as follows:

S.No.	out of 6 Positive Numbers	out of 8 Negative Numbers	Positive Numbers
1	4	0	${}^6C_4 \times {}^8C_0 = \frac{6\times5}{1\times2} \times 1 = 15$
2.	2	2	${}^6C_2 \times {}^8C_2 = \frac{6\times5}{1\times2} \times \frac{8\times7}{1\times2} = 420$
3.	0	4	${}^6C_0 \times {}^8C_4 = 1 \times \frac{8\times7\times6\times5}{1\times2\times3\times4} = 70$
			Total $= 505$

4

$$\text{Probability} = \frac{{}^6C_4 \times {}^8C_0 + {}^6C_2 \times {}^8C_2 + {}^6C_0 \times {}^8C_4}{{}^{14}C_4}$$

6

$$= \frac{15+420+70}{\frac{14\times13\times12\times11}{1\times2\times3\times4}}$$

$$= \frac{505\times4\times3\times2\times1}{14\times13\times12\times11}$$

$$= \frac{505}{1001} \quad \underline{\text{Ans}}$$

8

b. **An underground mine has 5 pumps installed for pumping out storm water, the probability of any one of the pumps failing during the storm is $\frac{1}{8}$. What is the probability that (a) at least 2 pumps will be working, (b) all the pumps will be working during a particular storm? (8)**

Answer:

(a) Probability of pump failing $= \frac{1}{8}$

Probability of pump working $= 1 - \frac{1}{8} = \frac{7}{8}$, $p = \frac{7}{8}$, $q = \frac{1}{8}$, $n = 5$

P(At least 2 pumps working) = P = (2 or 3 or 4 or 5 pumps working)

4

$$= P(2) + P(3) + P(4) + P(5)$$

$$= {}^5C_2 p^2 q^3 + {}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5 q^0$$

$$= 10\left(\frac{7}{8}\right)^2\left(\frac{1}{8}\right)^3 + 10\left(\frac{7}{8}\right)^3\left(\frac{1}{8}\right)^2 + 5\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right) + \left(\frac{7}{8}\right)^5$$

$$= \frac{1}{8^5}\left[10\times 49 + 10\times 343 + 5\times 2401 + 16807\right]$$

$$= \frac{1}{8^5}\left[490 + 3430 + 12005 + 16807\right]$$

$$= \frac{32732}{8^5} = \frac{8183}{8192} \quad \text{Ans} \qquad 8$$

(b) $P(\text{All the 5 pumps working}) = P(5)$

$$= {}^5C_5 p^5 q^0 \qquad 6$$

$$= \left(\frac{7}{8}\right)^5 = \frac{16807}{32768} \quad \text{Ans} \quad 8$$

Q.9 a. In a certain factory producing cycle tyres, there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using Poisson distribution, calculate the approximate number of lots containing no defective, one defective and two defective types respectively in a consignment of 10,000 lots. (8)

Answer:

$$p = \frac{1}{500}, \; n = 10$$

$$m = np = 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$$

$$P(r) = \frac{e^{-m} \cdot m^r}{r!} \qquad 4$$

(i) Probability of no. defective $= P(0) = \dfrac{e^{-0.02} \cdot (0.02)^0}{0!}$

$$= e^{-0.02} = 0.9802$$

Number of lots containing no defective $= 10{,}000 \times 0.9802$

$$= 9802 \text{ lots}$$

(ii) Probability of one defective $= P(1) = \dfrac{e^{-0.02}(0.02)^1}{1!}$

$$= 0.9802 \times 0.02 = 0.019604$$

6

Number of lots containing 1 defective

$$= 10{,}000 \times 0.019604 = 196 \text{ lots}$$

(iii) probability of two defectives $= P(2)$

$$= \frac{e^{-0.02}(0.02)^2}{\lfloor 2}$$

$$= 0.9802 \times 0.0002$$

$$= 0.00019604 \quad \underline{\text{Ans}}$$

4

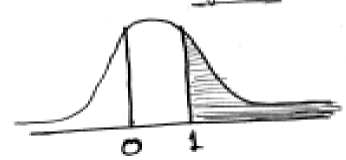
b. A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 gm. Find how many envelopes weighting (i) 2 gm or more, (ii) 2.1 gm or more, can be expected in a given packet of 1000 envelopes?
[Given: if t is the normal variable then, $\phi(0 \le t \le 1) = 0.3413$ and $\phi(0 \le t \le 2) = 0.477$]

(8)

Answer:

$\mu = 1.9$ gm, Variance $= 0.01$ gm

Fig



(i) $x = 2$ gm or more

$$z = \frac{x-\mu}{\sigma} = \frac{2-1.9}{0.1} = \frac{0.1}{0.1} = 1$$

$P(z > 1) =$ Area right to $z = 1$

$$= 0.5 - 0.3413 = 0.1587$$

4 Number of envelopes heavier than 2 gm in a lot of 1000

$$= 1000 \times 0.1587 = 158.7 = 159 \text{ (app.)}$$

Ans

(ii) $$z = \frac{2.1-1.9}{0.1} = \frac{0.2}{0.1} = 2$$

$P(z > 2) =$ Area right to $z = 2$

$$= 0.5 - 0.4772 = 0.0228$$

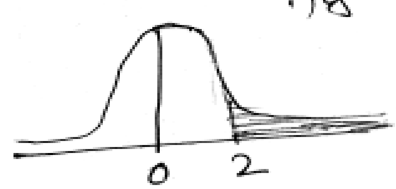
The number of envelopes heavier than 2.1 gm in a lot of 1000.

$$= 1000 \times 0.0228 = 22.8 = 23 \text{ (App.)}$$

Ans

Fig.

8



TEXT BOOKS

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II. A Text book of engineering Mathematics – N.P. Bali and Manish Goyal , 7th Edition 2007, Laxmi Publication(P) Ltd