Solutions

Q.2 a. If u = f(x, y), where x = Φ(t) and y = φ (t), show that $\frac{du}{dt} = \frac{\partial u}{dx}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$ **(8)**

Answer:

We have $u = f(x, y)$, $x = \varphi(t)$, $y = \psi(t)$

Let us give increment δt to t, and let corresponding increments of x, y and u be $\delta x, \delta y$ and δu.

$$\therefore \quad u + \delta u = f(x+\delta x, y+\delta y)$$

Subtracting $\delta u = f(x+\delta x, y+\delta y) - f(x, y)$

$$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y) \quad —(1)$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \times \frac{\delta x}{\delta t} + \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t} \quad —(2)$$

Taking limit as $\delta t \to 0$, so that $\delta x \to 0$, $\delta y \to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{Lt}\left[\underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \underset{\delta t \to 0}{Lt} \frac{\delta x}{\delta t}\right] + \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \underset{\delta t \to 0}{Lt} \frac{\delta y}{\delta t}$$

$$= \underset{\delta y \to 0}{Lt} \frac{\partial f(x, y+\delta y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \quad —(3)$$

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \quad —(4)$$

b. Expand f(x, y) = sin(xy) in powers of (x–1) and $\left(y - \frac{\pi}{2}\right)$ **up to the second degree terms. (8)**

Answer:

Taylor's expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{\lfloor 2}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}(a,b)\} + \dots \quad (1)$$

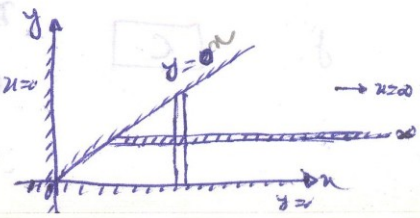
Substituting the values of $a = 1$, $b = \frac{\pi}{2}$, $f(a,b) = \sin(\frac{\pi}{2})$, $f_x(1, \frac{\pi}{2}) = \frac{\pi}{2}\cos(\frac{\pi}{2})$, etc

$$f(x,y) = \sin(xy) = \sin(1.\frac{\pi}{2}) + (x-1)\frac{\pi}{2}\cos(\frac{\pi}{2}) + (y-\frac{\pi}{2})1.\cos(\frac{\pi}{2}) + \frac{1}{\lfloor 2}(x-1)^2\frac{\pi^2}{4}(-1)$$
$$+ (x-1)(y-\frac{\pi}{2})\left[\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}\right] + \frac{1}{2}(y-\frac{\pi}{2})^2(-1).\sin\frac{\pi}{2}\dots$$
$$= 1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{\pi}{2}(x-1)(y-\frac{\pi}{2}) - \frac{1}{2}(y-\frac{\pi}{2})^2 + \dots \text{ upto second degree terms}$$

Q.3 a. Change the order of integration and then evaluate $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy \, dx$ **(8)**

Answer:

Ploting the boundries $x=0$, $x=\infty$, $y=0$, $y=x$, we get region of integration as shown in fig. To change order of integration, take strips parallel to x-axis. It moves parallel to itself form



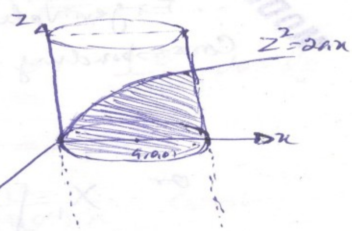
$y=0$ to $y=\infty$, keeping ends on $x=y$, $x=\infty$ so that changed order of integration is

$$\int_{y=0}^{y=\infty}\int_{x=y}^{x=\infty} x e^{-\frac{x^2}{y}} \, dx \, dy$$
$$= \int_0^\infty \left. -\frac{y}{2} e^{-\frac{x^2}{y}} \right|_{x=y}^{x=\infty} dy = \int_0^\infty \frac{y}{2} e^{-y} \, dy$$
$$= \frac{1}{2}\left[-y e^{-y}\Big|_0^\infty + \int_0^\infty 1.e^{-y} dy\right] = \frac{1}{2}\int_0^\infty e^{-y} dy = \frac{1}{2}$$

b. Find the volume enclosed by the cylinders $x^2+y^2 = 2ax$ and $z^2 = 2ax$ (8)

Answer:

Reqd volume bounded by the cylinders $x^2+y^2=2ax$ and $z^2=2ax$ is twice the volume shown.



$$\therefore V = 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} z\,dy\,dx$$

$$= 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \sqrt{2ax}\,dy\,dx = 2\int_0^{2a}\sqrt{2ax}\;2\sqrt{2ax-x^2}\,dx$$

$$= 4\sqrt{2a}\int_0^{2a}\sqrt{x}\sqrt{2ax-x^2}\,dx = 4\sqrt{2a}\int_0^{\pi/2}\sqrt{2a}\cdot\sin\theta\sqrt{4a^2\sin^2\theta-4a^2\sin^4\theta}\cdot 4a\sin\theta\cos\theta\,d\theta$$

(by putting $x=2a\sin^2\theta$)

$$= 64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta = 64a^3\cdot\frac{2}{15} = \frac{128}{15}a^3$$

Q.4 a. Show that the equations 3x + 3y + 2z = 1, x + 2y = 4, l0y + 3z = -2, 2x – 3y – z = 5 are consistent and solve them. (8)

Answer:

The given equation can be rewritten in matrix form as

$$\begin{bmatrix}3&3&2\\1&2&0\\0&10&3\\2&-3&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\4\\-2\\5\end{bmatrix}$$

Operate $R_1' = R_1-(R_2+R_4)$, $R_4' = R_4-2R_2$; $R_3' = R_3-R_1$, $R_3'' = \frac{R_3'}{6}$

$$\begin{bmatrix}0&4&3\\1&2&0\\0&10&3\\0&-7&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-8\\4\\-2\\-3\end{bmatrix},\quad \begin{bmatrix}0&4&3\\1&2&0\\0&1&0\\0&-7&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-8\\4\\1\\-3\end{bmatrix}$$

$R_1' = R_1-4R_3$, $R_2' = R_2-2R_3$, $R_4' = R_4+7R_3$; $R_1' = R_1+3R_4$

$$\begin{bmatrix}0&0&3\\1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-12\\2\\1\\4\end{bmatrix} \text{ or } \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\2\\1\\4\end{bmatrix}$$

It shows

Rank of coeff Matrix = Rank of Augmented Matrix = 3 = no. of Variables. Hence eqns are consistent.

Only 3 eqns are independent. The Sol is $x=2$, $y=1$, $z=-4$

b. Find the eigen values and eigen vectors of the matrix. (8)

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Answer:

Characteristic eqn. of the given matrix is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0, \quad \text{Solving } \lambda = 1, 1, 5$$

$$(\lambda-1)(\lambda-1)(\lambda-5) = 0$$

$\therefore$ Eigen values are $1, 1, 5$.

Corresponding to $\lambda = 1$, the eigen vectors (x_1, x_2, x_3) are given by

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{or} \quad x_1 + 2x_2 + x_3 = 0$$

or $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ are two independent eigen vectors corresponding to eigen values $\lambda = 1, 1$.

Corresponding to $\lambda = 5$, the eigen vector $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is given by

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{or} \quad \begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$$

Solving $x_1 = x_2 = x_3 = 1$. Hence eigenvector corresponding to eigen value $\lambda = 5$ is $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Q.5 a. Solve the differential equation $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + \sin 2x$ **(8)**

Answer:

A.E. of the given diff eqn. is

$m^3 + 2m^2 + m = 0$ or its roots are $m = 0, -1, -1$.

$\therefore$ CF is $= c_1 + (c_2 + c_3 x)e^{-x}$

P.I. is given by $= \dfrac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$

$$= \frac{1}{2^3 + 2.2^2 + 2} e^{2x} + \frac{1}{D(-4) + 2(-4) + D} \sin 2x$$

$$= \frac{1}{18} e^{2x} + \frac{1}{-3D - 8} \sin 2x = \frac{1}{18} - \frac{3D - 8}{(3D+8)(3D-8)} \sin 2x$$

$$= \frac{e^{2x}}{18} - \frac{1}{9(-4) - 64}(3.2\cos 2x - 8\sin 2x) = \frac{e^{2x}}{18} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$$

Hence complete sol. is

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x.$$

b. Solve the differential equation $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\{\log(1+x)\}$ **(8)**

Answer:

It is a Legendre's linear eqn..

Put $1+x = e^t$ or $t = \log(1+x)$ so that $(1+x)\frac{dy}{dx} = Dy$, $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given eqn. becomes

$$D(D-1)y + Dy + y = 4\cos t.$$

or $(D^2+1)y = 4\cos t$ where $D = \frac{d}{dt}$

Its C.F $= c_1\cos t + c_2\sin t$ because roots of A.E. are $\pm i$.

$$P.I = \frac{1}{D^2+1}4\cos t = 4t\frac{1}{2D}\cos t = 2t\int\cos t\,dt = 2t\sin t$$

Hence complete sol. is

$$y = c_1\cos t + c_2\sin t + 2t\sin t$$
$$= c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)]$$

Q.6 a. Use the method of false position to find the fourth root of 32 correct to three decimal places. (8)

Answer:

Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.*, a root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}f(x_0) = 2 + \frac{16}{65} = 2.2462 \quad ...(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

$\therefore$ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (i), we get $$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

$\therefore$ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (i), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770$, $x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

b. Apply Euler's method to solve for y at x = 0.1 from $\frac{dy}{dx} = x + y + xy, y(0) = 1$ taking step size h = 0.025 **(8)**

Answer:

By Euler's method

$$y_n = y_{n-1} + h f(x_0 + \overline{n-1}\, h, y_{n-1})$$

Take $x_0 = 0, y_0 = 1, h = 0.025, f(x, y) = x + y + xy$

$$y_1 = 1 + (0.025)[0 + 1] = 1.025$$

$$y_2 = 1.025 + (0.025)[(.025) + 1.025 + (.025)(1.025)]$$
$$= 1.025 + (0.025)(1.075625) = 1.05189$$

$$y_3 = 1.05189 + (.025)[.050 + 1.05189 + (.05)(1.05189)]$$
$$= 1.05189 + (.025)(1.15448) = 1.08075$$

$$y_4 = 1.08075 + (0.025)[0.075 + 1.08075 + (.075)(1.08075)] = 1.11154$$

Hence reqd. solution is $y = 1.11154$ at $x = 0.1$

Q.7 a. Obtain the series solution of $\frac{d^2y}{dx^2} + x^2 y = 0$ **(8)**

Answer:

(5) Here $x=0$ is ordinary point because $P_0(0) \neq 0$, therefore, let

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$
$$y'' = 2a_2 + 3\times 2\, a_3 x + 4\times 3\, a_4 x^2 + 5\times 4\, a_5 x^3 + \dots$$

Substituting these values in the given eqn., we get

$$\left(2a_2 + 3\times 2\, a_3 x + 4\times 3\, a_4 x^2 + 5\times 4\, a_5 x^3 + \dots\right) + x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right) = 0$$

equating to zero, coefficients of various powers of x,

$$a_2 = 0, \quad a_3 = 0, \quad 4\times 3\, a_4 + a_0 = 0, \quad 5\times 4\, a_5 + a_1 = 0, \quad 6\times 5\, a_6 + a_2 = 0$$

$$\therefore a_4 = -\frac{a_0}{4\times 3}, \quad a_5 = \frac{-a_1}{5\times 4}, \quad a_6 = -\frac{a_2}{6\times 5} = 0, \quad a_7 = -\frac{a_3}{7\times 6} = 0, \quad a_8 = -\frac{a_4}{8\times 7} = \frac{a_0}{8\times 7\times 4\times 3} \text{ etc}$$

Substituting in the given eqn.

$$y = a_0 + a_1 x + \frac{-a_0}{4\times 3} x^4 + \frac{-a_1}{5\times 4} x^5 + \frac{a_0}{8\times 7\times 4\times 3} x^8 + \frac{a_1}{9\times 8\times 5\times 4} x^9 + \dots$$
$$= a_0\left(1 - \frac{x^4}{4\times 3} + \frac{x^8}{8\times 7\times 4\times 3} - \frac{x^{12}}{12\times 11\times 8\times 7\times 4\times 3} + \dots\right) + a_1\left(x - \frac{x^5}{5\times 4} + \frac{x^9}{9\times 8\times 5\times 4} - \dots\right)$$

is required sol.

b. Show that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \beta(m+1, n+1)$ **(8)**

Answer:

$$\int_a^b (x-a)^m (b-x)^n \, dx = \int_0^{b-a} t^m (b-a-t)^n \, dt \quad \text{by putting } x-a=t$$
$$= \int_0^1 (b-a)^m z^m (b-a)^n (1-z)^n (b-a)\, dz, \quad \text{by putting } t=(b-a)z$$
$$= (b-a)^{m+n+1} \int_0^1 z^{(m+1)-1} (1-z)^{(n+1)-1} \, dz = (b-a)^{m+n+1} \beta(m+1, n+1)$$

Q.8 a. Prove that $\int_{-1}^{1} P_m(n)P_n(n)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$ **(4+4)**

Answer:

We know that $P_m(x)$ and $P_n(x)$ are solution of

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \quad — (1)$$

and

$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad — (2)$$

multiply (1) by v and (2) by u & subtract

$$(1-x^2)(u''v - v''u) - 2x(u'v - v'u) + [m(m+1) - n(n+1)]uv = 0$$

or $$\frac{d}{dx}\left[(1-x^2)(u'v - uv')\right] + (m-n)(m+n+1)uv = 0$$

Integrating w.r.t x from $x=-1$ to $x=1$,

$$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = \left|(1-x^2)(u'v - uv')\right|_{-1}^{+1} = 0$$

Hence $$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0, \text{ if } m \neq n$$

When $m=n$, we have from Rodrigues formula $P_n(x) = \dfrac{1}{\lfloor n\, 2^n}\dfrac{d^n}{dx^n}(x^2-1)^n$,

$$(\lfloor n\, 2^n)^2 \int_{-1}^{+1} P_n^2(x)\,dx = \int_{-1}^{+1} D^n(x^2-1)^n\, D^n(x^2-1)^n\,dx$$

$$= D^n(x^2-1)^n\, D^{n-1}(x^2-1)^n\Big|_{-1}^{+1} - \int_{-1}^{+1} D^{n+1}(x^2-1)^n\, D^{n-1}(x^2-1)^n\,dx \quad \text{(integrating by parts)}$$

$$= -\int_{-1}^{+1} D^{n+1}(x^2-1)^n\, D^{n-1}(x^2-1)^n\,dx \quad \because D^{n-1}(x^2-1)^n \text{ vanishes at } x = \pm 1$$

Continue integrating by parts

$$= (-1)^n \int_{-1}^{+1} D^{2n}(x^2-1)^n\,(x^2-1)^n\,dx = (-1)^n \int_{-1}^{+1} \lfloor 2n\,(x^2-1)^n\,dx$$

$$= 2\lfloor 2n \int_0^1 (1-x^2)^n\,dx = 2\lfloor 2n \int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta \quad \text{by putting } x = \sin\theta$$

$$= 2\lfloor 2n\,\frac{2n}{(2n+1)}\cdot\frac{2n-2}{(2n+1)}\cdot\ldots\cdot\frac{2}{3}\cdot = 2\lfloor 2n\,\frac{[2n(2n-2)\ldots 4\cdot 2]^2}{\lfloor 2n+1} = \frac{2}{2n+1}\left(2^n\lfloor n\right)^2$$

Hence $$\int_{-1}^{+1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

b. Prove that

$$J_4(n) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(n) + \left(1 - \frac{24}{x^2}\right) J_0(n) \qquad \textbf{(8)}$$

Answer:

We know from recurrence formula, $J_n = \frac{x}{2n}(J_{n-1} + J_{n+1})$ —(1)

$\therefore\ J_{n+1}(x) = \frac{2n}{x} J_n - J_{n-1}$

Putting $n = 1, 2, 3$, $J_2 = \frac{2}{x} J_1 - J_0$, $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$, $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

Substituting for J_3 and J_2, we get

$$J_4(x) = \frac{6}{x}\left(\frac{4}{x} J_2(x) - J_1(x)\right) - J_2(x) = \left(\frac{24}{x^2} - 1\right) J_2(x) - \frac{6}{x} J_1(x)$$

$$= \left(\frac{24}{x^2} - 1\right)\left(\frac{2}{x} J_1(x) - J_0(x)\right) - \frac{6}{x} J_1(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

Ans

Q.9 (For Current Scheme students i.e. AE51/AC51/AT51)

a. Solve the differential equation $(1 + x + y)^2 \frac{dy}{dx} = 1$ **(8)**

Answer:

Let $1 + x + y = z$ $\therefore\ 1 + \frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = \frac{dz}{dx} - 1$

Substituting in the given eqn., we get

$$z^2\left(\frac{dz}{dx} - 1\right) = 1 \quad \text{or} \quad \frac{dz}{dx} = 1 + \frac{1}{z^2} = \frac{1+z^2}{z^2}$$

Separating variables, $\frac{z^2}{1+z^2} dz = dx$

Integrating $\int \frac{z^2+1-1}{1+z^2} dz = \int dx + c$ or $z - \tan^{-1} z = x + c$

Hence reqd. sol. is $(1+x+y) - \tan^{-1}(1+x+y) = x + c$

or $y = \tan^{-1}(1+x+y) + k$ where $k = c - 1$, a new constant

b. Find the orthogonal trajectories of family of circles $x^2 + y^2 = 2ax$ (8)

Answer:

Given family of circles is $x^2+y^2=2ax$ ——— (i)

Diff. w.r.t. x, $2x+2y\frac{dy}{dx}=2a$ ——— (ii)

eliminating 'a', we get $2xy\frac{dy}{dx}=y^2-x^2$ ——— (iii)

$\therefore$ Diff. eqn. of orthogonal system is (Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$),

$-2xy\frac{dx}{dy}=y^2-x^2$ or $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$ ——— (iv)

To solve this homogeneous eqn., put $y=vx$, $\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ (iv) becomes $v+x\frac{dv}{dx}=\frac{2v}{1-v^2}$ or $x\frac{dv}{dx}=\frac{v+v^3}{1-v^2}$ or $\frac{1-v^2}{v(1+v^2)}dv=\frac{dx}{x}$

Integrating $\int\frac{1-v^2}{v(1+v^2)}dv=\log x+c$ or $\int\left[\frac{1}{v}-\frac{2v}{1+v^2}\right]dv=\log x+c$

or $\log v-\log(1+v^2)-\log x=c$ or $\frac{v}{(1+v^2)x}=\text{Const}$

Hence reqd. sol. is $\frac{y}{x}\frac{1}{\left(1+\frac{y^2}{x^2}\right)x}=\text{Const.}$ or $ky=(x^2+y^2)$

Q.9 (For New Scheme students i.e. AE101/AC101/AT101)

a. Find a Fourier Series to represent x^2 in the interval $(-\ell,\ell)$ (8)

Answer:

$f(x)=x^2$ is an even function in $(-l,l)$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$ ——— 1

$a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2l^2}{3}$ ——— 1

$a_n=\int_0^l x^2\cos\frac{n\pi x}{l}dx$

$=\frac{4l^2(-1)^n}{n^2\pi^2}$ ——— 4

$x^2=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left(\frac{\cos\pi x}{l}-\frac{\cos 2\pi x l}{2^2}+\frac{\cos 3\pi x l}{3^2}-\dots\right)$ ——— 1

b. Find the Fourier Cosine transform of e^{-x^2}. **(8)**

Answer:

$$F_c(e^{-x^2}) = \int_0^\infty e^{-x^2} \cos sx \, dx = I \quad \text{—— 1}$$

$$\frac{dI}{ds} = -\int_0^\infty x e^{-x^2} \sin sx \, dx. \quad \text{—— 1}$$

$$= -\frac{s}{2} \int_0^\infty e^{-x^2} \cos sx \, dx$$

$$= -\frac{s}{2} I$$

$$\log I = -\frac{s^2}{4} + \log c. \quad \text{—— 3}$$

$$\therefore I = c e^{-s^2/4}.$$

$$\int e^{-x^2} \cos sx \, dx = c e^{-s^2/4}.$$

Let $s = 0 \Rightarrow c = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. —— 1

$$\therefore I = \frac{\sqrt{\pi}}{2} e^{-s^2/4} \quad \text{—— 1}$$

$$\therefore F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}. \quad \text{—— 1}$$

TEXT BOOKS

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi

2. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd