Q.2 a. Check the validity of the following argument:- (8)
"If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect."

Answer:
Let p: "Labour market is perfect"; q : "Wages of all persons in a particular employment will be equal". Then the given statement can be written as **(7)**

[(p□q) □□~q] □□~p

Now, [(p□q) □□~q] □□~p = [(~p□q) □□~q] □□~p

= [(~p □□~q) □□(q □□~q)] □□~p

= [(~p □□~q) □□0] □□~p

= [~(p □□q)] □□~p

= ~[~(p □□q)] □□~p

= (p □□q) □□~p = (p □□~p) □□q = 1 □□q =1

A tautology. Hence the given statement is true.

3 marks for mutating
5 marks for validating

b. Let L be a distributive lattice. Show that if there exists an a with $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, then x = y. (8)

Answer:
In any distributive lattice L, for any three elements a, x, y of L, we can write

x = x□□(a □□x) **(7)**

= (x□□a) □□(x□x) [Distributive law]

= (a□□x) □□x [Commutative and Idempotent law]

= (a□□y) □□x [Given that (a□□x) = (a□□y)]

= (a □□y) □□(y □□x) [Distributive law]

= (a □□y) □□(y □□x) [Given that (a □□x) = (a □□y)]

= (y □□a) □□(y □□x) [Commutative law]

= y □□(a □□x) [Distributive law]

= y □□(a □□y) [Given that (a□□x) = (a□□y)]

=y.

Stepwise marking

Q.3 a. Solve the recurrence relation (8)
T(k) = 2 T (k-1), T(0) = 1

Answer:
The given equation can be written in the following form: **(8)**
tn - 2tn-1 = 0
Now successively replacing n by (n – 1) and then by (n – 2) and so on we get a set of equations.

The process is continued till terminating condition. Add these equations in such a way that all intermediate terms get cancelled. The given equation can be rearranged as

tn – 2tn–1 = 0

Stepwise marking

tn–1 – 2tn–2 = 0
tn–2 – 2tn–3 = 0
tn–3 – 2tn–4 = 0
..t
2 – 2t1 = 0
t1 – 2t0 = 0 [we stop here, since t0 = 1 is given]
Multiplying all the equations respectively by 2^0, 2^1, 2^2.........2^n-1and then adding them together, weget tn – 2^n t0 = 0
or, tn = 2^n

b. What are the different types of quantifiers? Explain in brief. (8)
Show that $(\exists x)(P(x) \wedge Q(x)) => (\exists x)P(x) \wedge (\exists x)Q(x)$

Answer: There are two quantifiers used in predicate calculus: Universal and existential.

Universal Quantifier: The operator, represented by the symbol $\forall$, used in predicate calculus to indicate that a predicate is true for all members of a specified set.
Some verbal equivalents are *"for each"* or *"for every"*. **(2+2)**

Existential Quantifier: The operator, represented by the symbol $\exists$, used in predicate calculus to indicate that a predicate is true for at least one member of a specified set.
Some verbal equivalents are *"there exists"* or *"there is"*.

Now, (□x) (P(x) □□Q(x)) => For some (x = c) (P(c) □□Q(c))

=>P(c) □□Q(c)

=> □x P(x) □□□x Q(x) **(4)**

Q.4 a. If f is a homomorphism from a commutative semigroup (S,*) onto a semigroup (T, *'), then show that (T, *') is also commutative. (8)

Answer:

The function f is said to be a ***homomorphism*** of S ***into*** T if **(7)**

f(a □□b) = f(a) □' f(b) □□a, b □□S

i.e. f preserves the composition in S and T.
It is given that (S, *) is commutative, so

a*b = b*a _ f(a*b) = f(b*a) □□a, b □□S

=> f(a) *' f(b) = f(b)*' f(a)
=>(T,*') is also commutative.

b. How many words of 4 letters can be formed with the letters a, b, c, d, e, f, g and h, when (8)
(i) e and f are not to be included
(ii) e and f are to be included

Answer:
There are 8 letters: a, b, c, d, e, f, g and h. In order to form a word of 4 letters, we have to find the number of ways in which 4 letters can be selected from 8 letters. Here order is important, so permutation is used.
(i) When **e** and **f** are **not** to be included, the available letters are 6 only. Thus we can select 4 out of six to form a word of length 4 in 6P4 ways =6! / (6-4)!= 360
Thus 360 words can be formed. **(4)**
(ii) When **e** and **f** are to be included, the available letters are all 8 letters. Thus we can select 4 out of 8 to form a word of length 4 in 8P4 ways = 8! / (8-4)! = 1680. Thus 1680 words can be formed. **(4)**

Q.5 a. Let x be the set of all programs of a given programming language. Let R the relation on x defined as P1 R P2 if P1 and P2 give the same output on all the inputs for which they terminate. Is R an equivalence relation? If not which property fails? (8)

Answer:
Ans: R is defined on the set X of all programs of a given programming language. Now let us test whether R is an equivalence relation or not
Reflexivity: Let P be any element of X then obviously P gives same output on all the inputs for which P terminates i.e. P R P. Thus R is reflexive.
Symmetry: Let P and Q are any two elements in X such that P R Q. Then P and Q give same boutput on all the inputs for which they terminate => Q R P =>R is symmetric. **(4)**
Transitivity: Let P, Q and S are any three elements in X such that P R Q and Q R S. It means P and Q gives the same out for all the inputs for which they terminate. Similar is the case for Q and S. It implies that there exists a set of common programs from which P, Q and S give the same output on all the inputs for which they terminate i.e. P R S => R is transitive.
Therefore R is an equivalence relation. **(4)**

b. Prove that for every positive integer n, $n^3 – n$ is divisible by 3. (8)

Answer:
Let P(n) be the proposition that for every positive integer n; n3 □ n is divisible by 3. We must show that P(1) is true and that the conditional statement P(k) =>P(k + 1) is true for k = 1; 2; 3; : : :
BASE STEP P(1) is true, because 1^3 - 1 = 0 is divisible by 3. **(2)**
INDUCTIVE STEP For the inductive hypothesis we assume that P(k) holds for an arbitrary positive integer k:That is, we assume that, For every positive integer k, k^3 - k is divisible by 3 ....(2)
Under this assumption, it must be shown that P(k + 1) is true, namely, that 3j[(k + 1)^3 - (k + 1)]
is also true
= (k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1
= k^3 - k + 3k^2 + 3k
= k3 - k + 3(k^2 + k).
Using (2) we know that k3-k is divisible by 3 for every positive integer k: Moreover

3(k^2 +k) is also a multiple of 3 for every positive integer k: Thus k^3 - k + 3(k^2 + k) is divisible by 3. Since we have completed both, the base step and the inductive step we have shown that P(n) is true for all positive integers n. **(6)**

Q.6 a. What are tautologies and contradiction? Prove that, for any propositions P, Q, R the following compound propositions are tautologies:

(i) $[(P \to Q) \wedge (P \to R)] \to (P \to R)$

(ii) $[P \to (Q \to R)] \to [(P \to Q) \to (P \to R)]$ **(8)**

Answer:

Page 57, Discrete Mathematical Structures by Trembly

b. Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$. **(8)**

Answer:

Let us find the truth table for $[(P \vee Q) \wedge (P \to R) \wedge (Q \to S)] \to (S \vee R)$. **(7)**

P Q R S	$(P \vee Q)$	$(P \to R)$	$(Q \to S)$	$(S \vee R)$	$[(P \vee Q) \wedge (P \to R) \wedge (Q \to S)] \to (S \vee R)$
0 0 0 0	0	1	1	0	1
0 0 0 1	0	1	1	1	1
0 0 1 0	0	1	1	1	1
0 0 1 1	0	1	1	1	1
0 1 0 0	1	1	0	0	1
0 1 0 1	1	1	1	1	1
0 1 1 0	1	1	0	1	1
0 1 1 1	1	1	1	1	1
1 0 0 0	1	0	1	0	1
1 0 0 1	1	0	1	1	1
1 0 1 0	1	1	1	1	1
1 0 1 1	1	1	1	1	1
1 1 0 0	1	0	0	0	1
1 1 0 1	1	0	1	1	1
1 1 1 0	1	1	0	1	1
1 1 1 1	1	1	1	1	1

Q.7 a. Consider the function f : N → N, where N is the set of natural numbers, defined by $f(n) = n^2 + n + 1$. Show that the function f is one-one but not onto. **(8)**

Answer:

In order to prove that f is one to one, it is required to prove that for any two integers n and m, if f(n) = f(m) then n = m. **(7)**

f(n) = f(m) ☐☐n^2+ n + 1 = m^2+ m + 1

☐☐n^2+ n = m^2+ m

☐☐n(n + 1) = m(m + 1)

☐☐n = m. Because product of consecutive natural numbers starting from m and n are equal iff m = n.
Next f is **not onto** because for any n (odd or even) n^2+ n + 1 is odd. This implies that there areeven elements in N that are not image of any element in N.

b. Using principles of inclusion and exclusion determine the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5 or 7. (8)

Answer:
Page No. 24 Dis. Math by C L I U Dr Mohrpeh

Q.8 a. Find the orders of the groups $U(Z_{10})$, $U(Z_{11})$, and $U(Z_{12})$, and describe their structure. (8)

Answer:
Solution The set U(Z10) is {1, 3, 7, 9}. In this case **(8)**
3^2 = 9, 3^3 = 7, 3^4 = 1,
so the group is a cyclic group of order 4.
U(Z11) = {1, 2, . . . , 10} and here
2^2 = 4, 2^3 = 8, 2^4 = 5, 2^5 = 10, 2^6 = 9, 2^7 = 7, 2^8 = 3, 2^9 = 6, 2^10 = 1,
So the group is a cyclic group of order 10.
U(Z12) = {1, 5, 7, 11}. Here every non-identity element has order 2:
5^2 = 1, 7^2 = 1, 11^2 = 1.
So in this case the group is the non-cyclic group of order 4, C2 × C2.

b. By finding a suitable generator, show that the multiplicative group of the field Z_{23} is cyclic. (8)

Answer:
In the absence of further information, the obvious method is to try the smallest numbers first. Trying 2, we get the powers
2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 1. **(8)**
Some non-zero elements of Z23, for example 10, are not in this list, so 2 is not a generator. Also, since 3 and 4 are in the list above, they cannot be generators (if 10 were a power of 3, it would also be a power of 2, since 3 = 2^8). So we must try 5, whose powers are:
5, 2, 10, 4, 20, 8, 17, 16, 11, 9, 22, 18, 21, 13, 19, 3, 15, 6, 7, 12, 14, 1.
This list is complete, so 5 is a generator and we have U(Z23) = <5> ≈ C22.

Q.9 a. Let G be the set of real numbers not equal to -1 and * be defined by a*b = a + b + ab. Prove that (G, *) is an abelian group. (8)

Answer:
Page No. 353 of Discrete Mathematics by Dr. D S C

b. Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ **(8)**

Answer:

(b) Ans: In order to prove this let x be any element of $A \cup B$, then **(7)**

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin A$) or ($x \in A$ and $x \in B$)

$\Leftrightarrow$ ($x \in A - B$) or ($x \in B - A$) or ($x \in A \cap B$)

$\Leftrightarrow x \in (A - B) \cup (B - A) \cup (A \cap B)$

This implies that

$A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$ and

$(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$

Thus $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

TEXT BOOK

I. Discrete Mathematical Structures, D. S. Chandrasekharaiah, Prism Books Pvt. Ltd., 2005