Q2 (a) Prove that cosA.cos(60°-A).cos(60°+A) = $\frac{1}{4}\cos 3A$ and deduce that $\cos 20^\circ.\cos 40^\circ.\cos 80^\circ = \frac{1}{8}$

Answer

We have,

$$\text{LHS} = \cos A\,[\cos(60^\circ+A)\cdot\cos(60^\circ-A)]$$
$$= \cos A\,[\cos^2 A - \sin^2 60^\circ]$$
$$= \cos A\,[\cos^2 A - \tfrac{3}{4}] \qquad \because \sin 60^\circ = \tfrac{\sqrt{3}}{2}$$
$$= \tfrac{1}{4}[4\cos^3 A - 3\cos A] = \tfrac{1}{4}\cos 3A = \text{RHS}.$$

Now putting $A = 20^\circ$, we get

$$\cos 20^\circ\cdot\cos(60^\circ-20^\circ)\cdot\cos(60^\circ+20^\circ) = \tfrac{1}{4}\cos 60^\circ$$

i.e. $\cos 20^\circ\cdot\cos 40^\circ\cdot\cos 80^\circ = \tfrac{1}{4}\cdot\tfrac{1}{2} = \tfrac{1}{8}$

Hence the Result.

Q2 (b) If A, B, C are the angle of a triangle, then prove that cosA + cosB + cosC = 1+ 4sinA/2. sinB/2 .sinC/2

Answer

Given $A+B+C=180^\circ$, $\therefore \frac{A+B}{2} = \frac{180}{2} - \frac{C}{2}$

$$\text{LHS} = (\cos A + \cos B) + \cos C$$
$$= 2\cos\left(\tfrac{A+B}{2}\right)\cdot\cos\left(\tfrac{A-B}{2}\right) + 1 - 2\sin^2 C/2$$
$$= 2\cos\left(90 - \tfrac{C}{2}\right)\cdot\cos\left(\tfrac{A-B}{2}\right) + 1 - 2\sin^2 C/2$$
$$= 2\sin\tfrac{C}{2}\cdot\cos\left(\tfrac{A-B}{2}\right) + 1 - 2\sin^2 C/2$$
$$= 1 + 2\sin C/2\left[\cos\left(\tfrac{A-B}{2}\right) - \sin\tfrac{C}{2}\right]$$

$\left[\because A+B+C=180^\circ,\ \frac{C}{2} = 90 - \frac{A+B}{2},\ \sin\frac{C}{2} = \sin\left(90^\circ - \frac{A+B}{2}\right) = \cos\left(\frac{A+B}{2}\right)\right]$

$$= 1 + 2\sin C/2\left[\cos\left(\tfrac{A-B}{2}\right) - \cos\left(\tfrac{A+B}{2}\right)\right]$$
$$= 1 + 2\sin C/2\cdot 2\sin\frac{\left(\frac{A-B}{2}+\frac{A+B}{2}\right)}{2}\cdot\sin\frac{\left(\frac{A+B}{2}-\frac{A-B}{2}\right)}{2}$$
$$= 1 + 4\cdot\sin\tfrac{A}{2}\cdot\sin\tfrac{B}{2}\cdot\sin\tfrac{C}{2}$$
$$= \text{RHS}.$$

$\left[\because \cos C - \cos D \rightarrow 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}\right]$

Q3 (a) Find the co-efficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$. Also find co-efficient of x^{-17}.

Answer

comparing $\left(x^4 - \frac{1}{x^3}\right)^{15}$ with $(X+a)^n$ we have,

$X = x^4$, $a = \frac{-1}{x^3}$, $n = 15$

$T_{r+1} = {}^nC_r X^{n-r} a^r = {}^{15}C_r (x^4)^{15-r} (-1)^r \left(\frac{1}{x^3}\right)^r$

$= {}^{15}C_r x^{60-4r} \cdot (-1)^r \cdot \frac{1}{x^{3r}} = {}^{15}C_r x^{60-7r} \cdot (-1)^r$

Put $60 - 7r = 32 \Rightarrow r = 4$

$T_5 = {}^{15}C_4 x^{32} \cdot (-1)^4$

Coefficient of x^{32} is $= {}^{15}C_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12}{4 \cdot 3 \cdot 2 \cdot 1} = 1365$ Ans

Again put $60 - 7r = -17 \Rightarrow r = 11$

$T_{12} = {}^{15}C_{11} x^{-17} \cdot (-1)^{11}$

Coefficient of x^{-17} is $= -{}^{15}C_{11} = \frac{-(15)(14)(13)(12)}{4 \cdot 3 \cdot 2 \cdot 1} = -1365$ Ans

Q3 (b) The sum of first three terms of a G.P. is 16 and the sum of the next three term is 128. Find the sum of the n^{th} term of GP.

Answer

b. Let a be the first term and r the common ratio of the G.P., then

$a + ar + ar^2 = 16$ —— (i)

and $ar^3 + ar^4 + ar^5 = 128$ —— (ii)

$\Rightarrow a(1+r+r^2) = 16$ and $ar^3(1+r+r^2) = 128$

$\Rightarrow \frac{\cancel{a} r^3 \cancel{(1+r+r^2)}}{\cancel{a} \cancel{(1+r+r^2)}} = \frac{\cancel{128}}{\cancel{16}}\ 8$

$\Rightarrow r^3 = 8 \Rightarrow 2 = r$

Putting the value of $r = 2$ in (i), we get $a = \frac{16}{7}$

$\therefore S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

$\Rightarrow S_n = \frac{16}{7}\left(\frac{2^n - 1}{2 - 1}\right) = \frac{16}{7}(2^n - 1)$ Ans.

Q4 (a) Show that, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$

Answer

Let $A = \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{vmatrix}$

$|A| = 1+\tan^2\theta$

Now, C_{11} = co-factor of $A_{11} = 1$

C_{12} = co-factor of $A_{12} = \tan\theta/2$

C_{21} = co-factor of $A_{21} = -\tan\theta/2$

C_{22} = co-factor of $A_{22} = 1$

$\therefore \quad A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A = \frac{1}{1+\tan^2\theta/2} \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

emaining of Q.4.a. 26

$$\text{RHS.} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \cdot A^{-1}$$

$$= \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \cdot \frac{1}{1+\tan^2\theta/2} \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2\theta/2} \begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} & \frac{-2\tan\theta/2}{1+\tan^2\theta/2} \\ \frac{2\tan\theta/2}{1+\tan^2\theta/2} & \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$= \text{RHS}$ Hence proved.

Q4 (b) Find the matrix A satisfying the equation $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} A \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer

Soln:

Let $B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

then the given eqn. becomes,

$BAC = I_2$

we have, $|B| = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1 \neq 0$

and $|C| = \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = 1 \neq 0$

Therefore, B and C are investible matrix such that,

$B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Now, $BAC = I_2$

$\Rightarrow B^{-1}(BAC)C^{-1} = B^{-1} I_2 C^{-1}$

$\Rightarrow (B^{-1}B)A(CC^{-1}) = B^{-1}C^{-1}$

$\Rightarrow I_2 A I_2 = B^{-1}C^{-1} \Rightarrow A = B^{-1}.C^{-1}$

$\Rightarrow A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ -16 & 25 \end{bmatrix}$ Ans.

Q5 (a) For what values of K are the three lines 4x + 7y-9 = 0, 5x + ky + 15 = 0 and 9x - y + 6 = 0 are concurrent.

Answer

2.5.a. The given lines are, $4x+7y-9=0$ — (i)

$5x+ky+15=0$ — (ii)

$9x-y+6=0$ — (iii)

Solving (i) and (iii) simultaneously, we get

$$\frac{x}{42-9} = \frac{y}{-81-24} = \frac{1}{-4-63}$$

$$x = -\frac{33}{67}, \quad y = \frac{105}{67}$$

Thus the point of intersection of (i) and (iii) is $\left(-\frac{33}{67}, \frac{105}{67}\right)$.

Since, the three lines are concurrent, this point of intersection lies on (ii), ie $5x+ky+15=0$

$\Rightarrow 5\left(-\frac{33}{67}\right) + k\left(\frac{105}{67}\right) + 15 = 0$

$\Rightarrow -165 + 105k + 1005 = 0 \Rightarrow 105k = -840 \Rightarrow k = \frac{-840}{105} = -8$

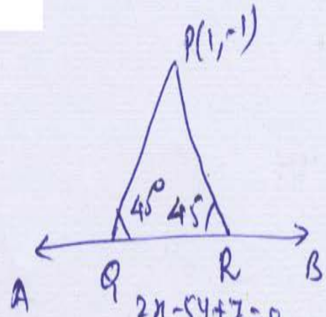
Hence, for $k = -8$, the given lines are concurrent.

Q5 (b) Find the equation of the two lines passing through the point (1, -1) and inclined at an angle of 45^o with the line 2x -5y +7 = 0

Answer

Soln: We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle α which the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$



Here, $x_1 = 1$, $y = -1$, $\alpha = 45$

m = Slope of the line $2x - 5y + 7 = 0$

So, $m = \frac{2}{5}$ (1) $\left[\text{using } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y}\right]$

So, the equation of the required lines are

$$y - (-1) = \frac{\frac{2}{5} + \tan 45^\circ}{1 - \frac{2}{5}\tan 45^\circ}(x-1) \text{ and } y - (-1) = \frac{\frac{2}{5} - \tan 45^\circ}{1 + \frac{2}{5}\tan 45^\circ}(x-1)$$

$$\Rightarrow y + 1 = \frac{\frac{2}{5} + 1}{1 - \frac{2}{5}(1)}(x-1) \quad \text{and} \quad y + 1 = \frac{\frac{2}{5} - 1}{1 + \frac{2}{5}(1)}(x-1)$$

$$\Rightarrow y + 1 = \frac{7}{3}(x-1) \quad \text{and} \quad y + 1 = \frac{-3}{7}(x-1)$$

$$\Rightarrow 3y + 3 = 7x - 7 \quad \text{and} \quad 7y + 7 = -3x + 3$$

$$\Rightarrow 7x - 3y - 10 = 0 \quad \text{and} \quad 3x + 7y + 4 = 0$$

Hence, the equation of the required lines are

$7x - 3y - 10 = 0$ and $3x + 7y + 4 = 0$ Ans.

Q6 (a) Find the equation of the circle which passes through the intersection of two circles, $x^2 + y^2 - 8x - 24y + 7 = 0$, and $x^2 + y^2 - 4x + 10y + 8 = 0$ and has it centre on the x-axis.

Answer

Sol. we have two circles,

$$x^2+y^2-8x-24y+7=0 \quad \text{—(i)}$$

$$x^2+y^2-4x+10y+8=0 \quad \text{—(ii)}$$

Let the equation of a circle passing through the pts. of intersection of (i) and (ii) be,

$$x^2+y^2-8x-24y+7+k(x^2+y^2-4x+10y+8)=0$$

$$(1+k)x^2+(1+k)y^2+(-8-4k)x+(-24+10k)y+7+8k=0$$

$$\Rightarrow x^2+y^2+\frac{-8-4k}{1+k}x+\frac{-24+10k}{1+k}y+\frac{7+8k}{1+k}=0 \quad \text{—(iii)}$$

The centre of the circle (iii) is $\left(\frac{4+2k}{1+k}, \frac{12-k}{1+k}\right)$,

The centre lies on x-axis, so, $[\because y=0]$

$$\frac{12-5k}{1+k}=0 \Rightarrow 12-5k=0 \Rightarrow k=\frac{12}{5}$$

Putting the value of $k=\frac{12}{5}$ in eq. (iii), we have

Q6.a. Sol.

$$x^2+y^2+\frac{-8-\frac{48}{5}}{1+\frac{12}{5}}x+\frac{-24+24}{1+\frac{12}{5}}y+\frac{7+\frac{96}{5}}{1+\frac{12}{5}}=0$$

$$\Rightarrow x^2+y^2-\frac{88}{17}x+\frac{131}{17}=0 \quad \text{— Ans}$$

Q6 (b) Find the centre, length of the axes, eccentricity, directrices, foci and the length of the latus rectum of the hyperbola $9x^2 - 16y^2 = 144$

Answer

Soln:

The given eqn. of the hyperbola can be written as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Here, $a^2 = 16$ and $b^2 = 9$

$\Rightarrow$ $a = 4$ and $b = 3$

$\therefore$ Transverse Axis $= 2a = 2 \times 4 = 8$

$\therefore$ Conjugate Axis $= 2b = 2 \times 3 = 6$

$\therefore$ Centre $= (0, 0)$

We know that, $b^2 = a^2(e^2 - 1)$

$\Rightarrow 9 = 16(e^2 - 1) \Rightarrow e = \sqrt{\frac{25}{16}} = \frac{5}{4}$

Directrices are $x = \frac{a}{e} \Rightarrow x = \frac{4}{\left(\frac{5}{4}\right)} \Rightarrow x = \frac{16}{5}$

and $x = -\frac{a}{e} \Rightarrow x = \frac{-4}{\left(\frac{5}{4}\right)} \Rightarrow x = -\frac{16}{5}$

The co-ordinates of foci are $(\pm ae, 0)$ i.e, $(\pm 4 \times \frac{5}{4}, 0)$ i.e, $(\pm 5, 0)$

Length of Latus-rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$ Ans

Q8 (a) Evaluate $\int \frac{2x}{(x^2+1)(x^2+2)} dx$

Answer

Soln:

Let $I = \int \frac{2x}{(x^2+1)(x^2+2)} dx$ —— (i)

On putting $x^2 = t$ so that $2x\,dx = dt$ in (i), we get,

$$I = \int \frac{dt}{(t+1)(t+2)}$$

Let $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$ —— (ii)

$\Rightarrow 1 \equiv A(t+2) + B(t+1)$

$1 \equiv A(-1+2) \Rightarrow A = 1 \quad (t = -1)$

$1 \equiv B(-2+1) \Rightarrow B = -1 \quad (t = -2)$

$$\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} - \frac{1}{t+2}$$

$$\int \frac{1}{(t+1)(t+2)} dt = \int \frac{1}{t+1} dt - \int \frac{1}{t+2} dt$$

$$= \log(t+1) - \log(t+2) = \log \frac{t+1}{t+2}$$

$$= \log \frac{x^2+1}{x^2+2}$$ Ans.

Q8 (b) Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Answer

Soln:

$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

then

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2 x} dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x} - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x}$$

Put $\cos x = t$, $-\sin x \, dx = dt$

when $x = \pi$, $t = -1$

when $x = 0$, $t = 1$.

$$2I = -\int_{+1}^{-1} \frac{\pi \, dt}{1+t^2} = \pi \int_{-1}^{+1} \frac{dt}{1+t^2} = \pi \left[\tan^{-1} t\right]_{-1}^{+1}$$

$$= \pi\left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

$$= \pi\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4} \quad \text{Ans.}$$

Q9 (a) Solve the initial value problem (1 + x²) $\frac{dy}{dx} + 2xy - 4x^2 = 0$, **when y(0) = 0.**

Answer

Soln: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{4x^2}{1+x^2} \quad \text{—(i)}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

we have $I.F. = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1+x^2$

multiplying both sides of (i) by I.F. $= (1+x^2)$, we get

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Integrating w.r.t. x, we get

$y(1+x^2) = \int 4x^2 dx + c$ $\quad [\text{using } y(I.F.) = \int Q(I.F.)dx + c]$

or $y(1+x^2) = \frac{4}{3}x^3 + c$

It is given that $y = 0$ when $x = 0$, So, $0 = 0 + c \Rightarrow c = 0$

Hence, $y(1+x^2) = \frac{4x^3}{3}$ is the required solution. Ans

Q9 (b) Solve x(y-x) $\frac{dy}{dx} = y(y+x)$

Answer

Soln:

We have, $\frac{dy}{dx} = \frac{y(y+x)}{x(y-x)}$

The given differential equation is a homogeneous differential equation. Substituting $y = vx$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \text{ we get}$$

$$v + x\frac{dv}{dx} = \frac{vx(vx+x)}{x(vx-x)}$$

$$\text{or} \quad v + x\frac{dv}{dx} = \frac{v(v+1)}{v-1}$$

$$\text{or} \quad x\frac{dv}{dx} = \frac{v^2+v-v^2+v}{v-1} = \frac{2v}{v-1}$$

By separating variable, we get $\frac{v-1}{v}\,dv = 2\frac{dx}{x}$

on integrating the variable, we get $\frac{v-1}{v}\,dv = 2\frac{dx}{x}$

on integrating $\int\left(1-\frac{1}{v}\right)dv = 2\int\frac{1}{x}\,dx$

$$\text{or} \quad v - \log v = 2\log x + c$$

$$\text{or} \quad \frac{y}{x} - \log\left(\frac{y}{x}\right) = \log x^2 + c$$

$$\text{or} \quad \frac{y}{x} - \log xy = c. \quad \text{Ans}$$

or $\frac{y}{\;}$

Text Books

1. Engineering Mathematics, H K Dass, S Chand & Company Ltd., New Delhi 2010.

2. A textbook of comprehensive mathematics class XI, Parmanand Gupta, Laxmi Publication (P) Ltd., New Delhi.