Q2 (a) The signals $x_1(t) = 10\cos(100\pi t)$ and $x_2(t) = 10\cos(50\pi t)$ are both sampled at $f_s = 75$ Hz. Show that the two sequences of samples so obtained are identical.

Answer

Ans. If the signal is sampled at $f_s = 75$ Hz, the discrete-time signal is

$$x_1(t)|_{t=nT_s} = x_1(nT_s) = x_1(n) = 10\cos(100\pi nT_s) = 10\cos\left(\frac{100\pi}{75}n\right) = 10\cos\left(\frac{4\pi}{3}n\right)$$
$$= 10\cos\left(2n\pi - \frac{2\pi}{3}n\right) = 10\cos\left(\frac{2\pi}{3}n\right)$$

Similarly, we have

$$x_2(t)|_{t=nT_s} = x_2(nT_s) = x_2(n) = 10\cos(50\pi nT_s) = 10\cos\left(\frac{50\pi}{75}n\right) = 10\cos\left(\frac{2\pi}{3}n\right) = x_1(n)$$

Q2 (b) Define quantization and quantization error? Derive signal to quantization noise ration for sinusoidal signals.

Answer

Ans. Quantization is a process of converting a continuous amplitude sample into a discrete amplitude sample. In other words, the process of converting a discrete-time continuous-amplitude signal by expressing each sample value as a finite (instead of an infinite) number of digits, is called quantization. The error introduced in representing the continuous-valued signal by a finite set of discrete value levels is called quantization error.

Let $x_q(n)$ denote the sequence of quantized samples at the output of the quantizer. The quantization error is a sequence $e_q(n)$ defined as the difference between the quantized value and the actual sample value. Thus

$$e_q(n) = x_q(n) - x(n)$$

The quantization error $e_q(n)$ in rounding is limited to the range of $-\Delta/2$ to $\Delta/2$, that is,

$$\frac{\Delta}{2} \le e_q(n) \le \frac{\Delta}{2}$$

where Δ is the step size. Consider a continuous time sinusoidal signal

$$x_a(t) = A\cos(\Omega_o t)$$

The quantization error is given by

$$e_q(t) = x_a(t) - x_q(t)$$

Let τ denotes the time that $x_a(t)$ stays within the quantization levels. The mean square error power P_q is

$$P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)dt = \frac{1}{\tau}\int_0^{\tau} e_q^2(t)dt$$

Since $e_q(t) = \frac{\Delta}{2\tau}t$, $-\tau \le t \le \tau$, we have

$$P_q = \frac{1}{\tau}\int^{\tau}\left(\frac{\Delta}{2\tau}\right)^2 t^2 dt = \frac{\Delta^2}{12}$$

If the quantizer has b bits of accuracy and the quantizer covers the entire range $2A$, the quantization step is $\Delta = 2A/2^b$. Hence

$$P_q = \frac{A^2/3}{2^{2b}}$$

The average power of the signal is given by

$$P_x = \frac{A^2}{2}$$

The signal to quantization noise ration is given by

$$SQNR = \frac{P_x}{P_q} = \frac{3}{2}2^{2b}$$

Q3 (a) A discrete-time causal LTI system has the system function

$$H(z) = \frac{(1+0.2z^{-1})(1-9z^{-2})}{1+0.81z^{-2}}$$

(i) Is the system stable?

(ii) Find expressions for a minimum-phase system $H_{min}(z)$ and an all pass system $H_{ap}(z)$ such that $H(z) = H_{min}(z)H_{ap}(z)$

Answer

Ans. (i) Consider the given system function

$$H(z) = \frac{(1+0.2z^{-1})(1-9z^{-2})}{(1+0.81z^{-2})}$$

For poles

$$1+0.81z^{-2} = 0$$
$$z^2 = -0.81 = j^2 0.81$$
$$z = \pm j0.9$$

The poles $z = \pm j0.9$ are inside the unit circle, so the system is stable.
(ii) Consider the given system function

$$H(z) = \frac{(1+0.2z^{-1})(1-9z^{-2})}{(1+0.81z^{-2})} = \frac{(1+0.2z^{-1})}{(1+0.81z^{-2})}(1-9z^{-2})$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include $(9-z^{-2})$ in both parts of the equation above the first part will be minimum-phase and the second will become allpass.

$$H(z) = \underbrace{\left(\frac{(1+0.2z^{-1})(9-z^{-2})}{(1+0.81z^{-2})}\right)}_{H_{min(z)}} \underbrace{\left(\frac{(1-9z^{-2})}{(9-z^{-2})}\right)}_{H_{ap}(z)}$$

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

Q3 (b) A nonminimum-phase causal signal x(n) has z-transform

$$X(z)=\frac{\left(1-\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{5}{3}z^{-1}\right)}{(1-z^{-1})^2\left(1-\frac{1}{4}z^{-1}\right)}$$

For what values of the constant β will the signal $y(n)=\beta^n x(n)$ be minimum-phase?

Answer

For what values of the constant β will the signal $y(n)=\beta^n x(n)$ be minimum-phase?

Ans. Given that

$$y(n)=\beta^n x(n)$$

$$Y(z)=X\left(\frac{z}{\beta}\right)$$

$$=\frac{\left(1-\frac{3}{2}\beta z^{-1}\right)\left(1+\frac{1}{3}\beta z^{-1}\right)\left(1+\frac{5}{3}\beta z^{-1}\right)}{(1-\beta z^{-1})^2\left(1-\frac{1}{4}\beta z^{-1}\right)}$$

In order for $Y(z)$ to be minimum-phase, all of the poles and zeros must be inside the unit circle. Poles are at $\beta, \beta, \frac{1}{4}\beta$ and zeros are at $\frac{3}{2}\beta, -\frac{1}{3}\beta, -\frac{5}{3}\beta$. Because the zero of $X(z)$ which is the farthest from the unit circle is at $z=-\frac{5}{3}$, $y(n)$ will be minimum-phase if $|\beta|<\frac{3}{5}$.

Q4 (a) Explain function declaration, function definition and function cell using a suitable example. What is function prototype? Find the 10-point inverse DFT of $X(k)=1+2\delta(k)$.

Answer

Ans. Given that $X(k)=1+2\delta(k)$, $0\le k\le 9$. Here $N=10$. We know that the inverse DFT of a constant is a unit impulse function.

$$\delta(n)\overset{\text{DFT}}{\underset{10}{\longleftrightarrow}}1, \qquad 0\le k\le 9$$

Similarly, the DFT of a constant is an impulse function.

$$1\overset{\text{DFT}}{\underset{10}{\longleftrightarrow}}10\delta(k)$$

$$\frac{1}{5}\overset{\text{DFT}}{\underset{10}{\longleftrightarrow}}2\delta(k)$$

Therefore it follows that

Q4 (b) Consider the length-12 sequence, defined for $0 \le n \le 11$**,**
$x(n) = \{\underline{3}, -1, 2, 4, -3, -2, 0, 1 - 4, 6, 2, 5\}$ **with a 12-point DFT given by** $X(k)$**,** $0 \le k \le 11$**. Evaluate the following functions of** $X(k)$ **without computing the DFT:**

(i) $X(0)$ **(ii)** $X(6)$ **(iii)** $\sum_{k=0}^{11} X(k)$ **(iv)** $\sum_{k=0}^{11} e^{-j4\pi k/6} X(k)$ **(v)** $\sum_{k=0}^{11} |X(k)|^2$

Answer

Ans. Given that $N = 12$.

(a)

$$X(0) = \sum_{n=0}^{N-1} x(n) = \sum_{n=0}^{11} x(n) = 13$$

(b)

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} (-1)^n x(n)$$

$$X(6) = \sum_{n=0}^{11} (-1)^n x(n) = 3 + 1 + 2 - 4 - 3 + 2 + 0 - 1 - 4 - 6 + 2 - 5 = -13$$

(d) Given that

$$\sum_{k=0}^{11} e^{-j\frac{4\pi}{6}k} X(k) = \sum_{k=0}^{11} e^{-j\frac{8\pi}{12}k} X(k) = \sum_{k=0}^{11} e^{-j\frac{2\pi}{12}4k} X(k)$$

Using the definition of IDFT $(N = 12)$, we obtain

$$\frac{1}{12}\sum_{k=0}^{11} X(k) e^{j\frac{2\pi}{12}nk} = x(n)$$

$$\sum_{k=0}^{11} X(k) e^{j\frac{2\pi}{12}nk} = 12x(n), \qquad \text{at } n = -4, \text{ we get}$$

$$\sum_{k=0}^{11} X(k) e^{-j\frac{2\pi}{12}4k} = 12x(\langle -4\rangle_{12}) = 12x(12-4) = 12x(8) = 12 \times -4 = -48$$

(e) Using Parseval's relation, we have

$$\sum_{k=0}^{11} |X(k)|^2 = 12\sum_{n=0}^{11} |x(n)|^2 = 12[9 + 1 + 4 + 16 + 9 + 4 + 0 + 1 + 16 + 36 + 4 + 25] = 1500$$

Q5 (a) What is FFT? Develop DIT-FFT algorithm for $N = 8$ and draw signal flow graph.

Answer

Ans. FFT is an algorithm to compute the DFT with reduced computation. The N-point DFT of sequence $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$$

Breaking $x(n)$ into its even and odd numbered values, we obtain

$$X(k) = \sum_{n=0, n\,\text{even}}^{N-1} x(n) W_N^{kn} + \sum_{n=0, n\,\text{odd}}^{N-1} x(n) W_N^{kn}$$

Substituting $n = 2r$ for n even and $n = 2r+1$ for n odd, we have

$$\begin{aligned} X(k) &= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k} \\ &= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{\frac{N}{2}}^{rk} \\ X(k) &= G(k) + W_N^k H(k), \quad 0 \le k \le N-1 \end{aligned}$$

where $G(k)$ and $H(k)$ are the $\frac{N}{2}$-point DFTs of the even and odd numbered sequences respectively. $G(k)$ and $H(k)$ are periodic with period $\frac{N}{2}$. Therefore,

$$\begin{aligned} X(k) &= G(k) + W_N^k H(k) && 0 \le k \le \frac{N}{2} - 1 \\ X\left(k + \frac{N}{2}\right) &= G\left(k + \frac{N}{2}\right) + W_N^{k+N/2} H\left(k + \frac{N}{2}\right) && 0 \le k \le \frac{N}{2} - 1 \end{aligned}$$

since $G\left(k + \frac{N}{2}\right) = G(k)$, $H\left(k + \frac{N}{2}\right) = H(k)$, and $W_N^{k+N/2} = -W_N^k$, we obtain

$$\begin{aligned} X(k) &= G(k) + W_N^k H(k) && 0 \le k \le \frac{N}{2} - 1 \\ X\left(k + \frac{N}{2}\right) &= G(k) - W_N^k H(k) && 0 \le k \le \frac{N}{2} - 1 \end{aligned}$$

The above process may be continued by expressing each of the two $\frac{N}{2}$-point DFTs, $G(k)$ and $H(k)$ as a combination of two $\frac{N}{4}$-point DFTs. Each of the $\frac{N}{2}$-point DFTs is computed by breaking each of the sum in two $\frac{N}{4}$-point DFTs, which is then combined to give the $\frac{N}{2}$-point DFTs.

By splitting the DFT into its even and odd parts we have reduced the operation count from N^2 (for a DFT of length N) to $2(N/2)^2$ (for two DFTs of length $N/2$). The cost of the splitting is that we need an additional $O(N)$ operations to multiply by the twiddle factor W_N^k and recombine the two sums.

We can repeat the splitting procedure recursively $\log_2 N$ times until the full DFT is reduced to DFTs of single terms. The DFT of a single value is just the identity operation, which costs nothing. However since $O(N)$ operations were needed at each stage to recombine the even and odd parts the total number of operations to obtain the full DFT is $O(N \log_2 N)$.

Q5 (b) Let $x(n)$ be a real-valued N-point sequence $\left(N = 2^m\right)$. Develop a method to compute an N-point DFT $X'(k)$, which contains only the odd harmonics [i.e., $X'(k) = 0$ if k is even] by using only a real $\frac{N}{2}$-point DFT.

Answer

b. Let $x(n)$ be a real-valued N-point sequence ($N = 2^m$). Develop a method to compute an N-point DFT $X'(k)$, which contains only the odd harmonics [i.e., $X'(k) = 0$ if k is even] by using only a real $\frac{N}{2}$-point DFT. (8)

Ans. By definition

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le n \le N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

Let

$$X'(k) = X(2k+1), \quad 0 \le k \le \frac{N}{2} - 1$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{(2k+1)n}$$

$$+ \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{(2k+1)\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{2kn} W_N^n$$

$$+ \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^n W_N^{2kn} W_N^{Nk} W_N^{N/2}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n W_{N/2}^{kn}$$

let $g(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n$, then

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{N/2}^{kn}$$

First form the sequence $g(n)$, and then take its $\frac{N}{2}$-point DFT to get odd harmonics of $X(k)$.

Q6 (a) Discuss the factors that influence the choice of structure for realization of a LTI system.

Answer

Ans. The block diagram representation of a system (or filter) is termed as a *realization* of the system or, equivalently as a *structure* for realizing the system. A given transfer function $H(z)$, can be realized by several structures and

they are all equivalent in the sense that they realize the same transfer function under infinite precision of the coefficients. The major factors that influence the choice of a specific structure are computational complexity, memory requirements, and finite-word-length effects.

1. The meaning of *computational complexity* is the requirement of arithmetic operations (multiplications, and additions) to compute the output $y(n)$.
2. The meaning of *memory requirements* is the number of memory locations required to store the past inputs, past outputs, system coefficients, and any intermediate computed values.
3. The real hardware has a finite number of bits representing the past inputs, past outputs, system coefficients,and any intermediate computed values. The effect of quantization (rounding or truncation) in the multiplications and additions of signal values depends on the type of representation of binary numbers, whether they are in fixed form or floating form, or whether they are in sign magnitude or 2's-complement form. The effects of all these finite values for the number of bits used in hardware implementation is commonly called *finite-word-length effects.*

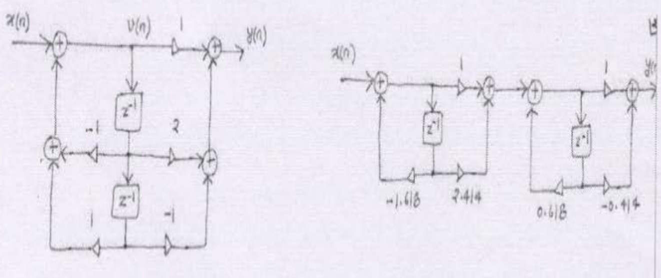

Fig. 1. Fig. 1(a) and (b)

Q6 (b) Obtain two canonical realizations of the system function:

$$H(z) = \frac{1 + 2z^{-1} - z^{-2}}{1 + z^{-1} - z^{-2}}$$

Answer

b. Obtain two canonical realizations of the system function,

$$H(z) = \frac{1+2z^{-1}-z^{-2}}{1+z^{-1}-z^{-2}}$$

Ans. A realization is said to be canonic if the number of delay elements is equal to the order of the filter. Consider the given system function

$$H(z) = \frac{1+2z^{-1}-z^{-2}}{1+z^{-1}-z^{-2}}$$

$$H(z) = \underbrace{\left[\frac{V(z)}{X(z)}\right]}_{\text{poles}}\underbrace{\left[\frac{Y(z)}{V(z)}\right]}_{\text{zeros}} = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1+z^{-1}-z^{-2}}$$

$$v(n) = -v(n-1) + v(n-2) + x(n)$$

and

$$H_2(z) = \frac{Y(z)}{V(z)} = 1 + 2z^{-1} - z^{-2}$$

direct form II realization is show in Fig. 1(a).

ow we realize $H(z)$ in cascade form. Again, consider the given system function

$$H(z) = \frac{1+2z^{-1}-z^{-2}}{1+z^{-1}-z^{-2}}$$

$$= \frac{1+2z^{-1}-z^{-2}+z^{-2}-z^{-2}}{1+z^{-1}-z^{-2}+\frac{1}{4}z^{-2}-\frac{1}{4}z^{-2}}$$

$$= \frac{\left(1+2z^{-1}+z^{-2}\right)-2z^{-2}}{\left(1+z^{-1}+\frac{1}{4}z^{-2}\right)-\frac{5}{4}z^{-2}}$$

$$= \frac{\left(1+z^{-1}\right)^2-\left(\sqrt{2}z^{-1}\right)^2}{\left(1+\frac{1}{2}z^{-1}\right)^2-\left(\frac{\sqrt{5}}{2}z^{-1}\right)^2}$$

$$= \frac{\left(1+z^{-1}+\sqrt{2}z^{-1}\right)\left(1+z^{-1}-\sqrt{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}+\frac{\sqrt{5}}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}-\frac{\sqrt{5}}{2}z^{-1}\right)}$$

$$H(z) = \frac{(1+2.414z^{-1})(1-0.414z^{-1})}{(1+1.618z^{-1})(1-0.618z^{-1})}$$

$$H(z) = \left[\frac{1+2.414z^{-1}}{1+1.618z^{-1}}\right]\left[\frac{1-0.414z^{-1}}{1-0.618z^{-1}}\right]$$

$$H(z) = H_1(z)H_2(z)$$

e $H_1(z)$ and $H_2(z)$ are first-order sections. The cascade form realization of $H(z)$ is shown in Fig. 1(b). This cascade employs two delay elements, which is equal to the order of the filter.

Q7 (a) Using a rectangular window, design a lowpass filter with a passband gain of unity, cutoff frequency of 1000 Hz and working at a sampling frequency of 5 KHz. Take the length of the impulse response as 7.

Answer

Ans. Given that gain=1, cutoff frequency $F_c = 1000$ Hz, sampling frequency $F_s = 5000$ Hz, and length $M = 7$. The digital cutoff frequency is given by $\omega_c = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1000}{5000} = 0.4\pi$. The frequency response of an ideal lowpass filter is given by

$$H_d(e^{j\omega}) = \text{rect}\left(\frac{\omega}{2\omega_c}\right) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

Its impulse response $h_d(n)$ is found using the inverse DTFT to give

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_c}^{\omega_c} = \frac{1}{\pi n}\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)$$

$$h_d(n) = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\left(\frac{\sin \omega_c n}{\omega_c n}\right) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\sin \omega_c n}{\pi n} & |n| > 0 \end{cases}$$

$$h_d(n) = \begin{cases} 0.4 & n = 0 \\ \frac{\sin 0.4\pi n}{\pi n} & |n| > 0 \end{cases}$$

$$h_d(n) = \{-0.0624,\ 0.0935,\ 0.3027,\ \underset{\uparrow}{0.4},\ 0.3027,\ 0.0935,\ -0.0624\}$$

The rectangular window is given by

$$w(n) = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

The impulse response of the length-7 FIR filter is given by

$$h(n) = h_d(n)w(n) = \{-0.0624,\ 0.0935,\ 0.3027,\ \underset{\uparrow}{0.4},\ 0.3027,\ 0.0935,\ -0.0624\}$$

Q7 (b) Explain the mapping of s-plane to z-plane using bilinear transformation with respect to IIR filter design.

Answer

Ans. Bilinear transformation is a one-to-one mapping from the s-domain to the z-domain. That is, the bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Also, the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain. The bilinear transformation is obtained by using the trapezoidal formula for numerical integration. Let the system function of the analog filter be

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$
$$sY(s) + aY(s) = bX(s)$$

Taking the inverse Laplace transform

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Integrating the above equation between the limits $(nT - T)$ and nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt} dt + a \int_{nT-T}^{nT} y(t)dt = b \int_{nT-T}^{nT} x(t)dt$$

The trapezoidal rule for numeric integration is given by

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}[a(nT) - a(nT-T)]$$

Using the above two equations, we get

$$[y(nT) - y(nT-T)] + \frac{aT}{2}[y(nT) - y(nT-T)] = \frac{bT}{2}[x(nT) - x(nT-T)]$$
$$[y(n) - y(n-1)] + \frac{aT}{2}[y(n) - y(n-1)] = \frac{bT}{2}[x(n) - x(n-1)]$$

Taking z-transform, the system function of the digital filter is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

Comparing $H(s)$ and $H(z)$, we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

The general characteristic of the mapping can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation, we get

$$s = \sigma + j\Omega = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right) = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

Therefore,

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right)$$
$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

From above equations it can be noted that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. thus, the left half of the s-plane maps inside the unit circle in the z-plane and the transformation results in a stable digital filter. For $r = 1$, we get

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

From above equation it is evident that the mapping is nonlinear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the nonlinearity of the arc tangent function and usually called as frequency warping.

Q8 (a) Consider a sequence $x(n) = \{\underline{1}, 2, 3, 4\}$ its DFT is given by $x(k) = \{\underline{10}, -2 + j2, -2, -2 - j2\}$. The sampling rate is 10 Hz.

(i) Determine the sampling period, time index and sampling time instant for a discrete time sample x(3) in time domain.

(ii) Determine the frequency resolution, frequency bin number and frequency for each of the DFT coefficients X(1) and X(3) in frequency domain.

Answer

Ans. (i) In time domain, we have the sampling period calculated as

$$T_s = \frac{1}{f_s} = \frac{1}{10} = 0.1 \text{ second}$$

For data $x(3)$, the time index is $n = 3$ and the sampling time instant is determined by

$$t = nT_s = 3 \times 0.1 = 0.3 \text{ seconds}$$

(ii) In frequency domain, since the total number of DFT coefficients if four (i.e., $N = 4$), the frequency resolution is determined by

$$\Delta f = \frac{f_s}{N} = \frac{10}{4} = 2.5 \text{ Hz}$$

The frequency bin number for $X(1)$ is $k = 1$ and its corresponding frequency is determined by

$$f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5 \text{ Hz}$$

Similarly, for $X(3)$ and $k = 3$,

$$f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5 \text{ Hz}$$

Note that $k = 3$ is equivalent to $k - N = 3 - 4 = -1$, and $f = 7.5$ Hz is also equivalent to the frequency $f = \frac{-1\times 10}{4} = -2.5$ Hz, which corresponds to the negative side spectrum.

Q8 (b) Write technical note on time-dependent Fourier transform.

Answer

Ans. The DFT can be employed for the spectral analysis of a finite-length signal composed of sinusoidal components as long as the frequency, amplitude and phase of each sinusoidal component are time-invariant and independent of signal length. There are practical situations (e.g. radar, sonar, speech and data communication signals) where the signal to be analyzed is instead nonstationary, for which these signal parameters are time-varying. An example of such a time-varying signal is the chirp signal given by

$$x(n) = \cos(\omega_o n^2)$$

Note that the instantaneous frequency of $x(n)$ is given by $2\omega_o n$, which is not a constant but increases linearly with time. A description of such a signal in the frequency-domain using a simple DFT of the complete signal will provide misleading results. To get around the time varying nature of the signal parameters, an alternative approach would be to segment the sequence into a set of subsequences of short length. If the subsequence length is reasonably small, it can be assumed to be stationary for practical purposes. As a result, the frequency-domain description of a long sequence is given by a set of short-length DFTs, i.e. a short-time Fourier transform.

The short-time Fourier transform (STFT), also known as the time-dependent Fourier transform, of a sequence $x(n)$ is defined by

$$X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x(n+m)w(m)e^{-j\omega m}$$

where $w(n)$ is a window sequence. The function of the window is to extract a finite-length portion of the sequence $x(n)$ such that the spectral characteristics of the section extracted are approximately stationary over the duration of the window.

The STFT is a function of two variables: time n which is discrete and the frequency variable ω, which is continuous. The STFT $X_{\text{STFT}}(e^{j\omega}, n)$ is a periodic function of ω with a period 2π.

In most applications, the magnitude of the STFT is of interest. The display of the magnitude of the STFT is usually referred to as the spectrogram. However, since the STFT is a function of two variables, the display of its magnitude would normally require three dimensions. Often, it is plotted in two dimensions, with the magnitude represented by the darkness of the plot. Here, the white areas the zero-valued magnitudes while the gray areas represent nonzero magnitudes, with the largest magnitudes being shown in black. In the STFT magnitude display, the vertical axis represents the frequency variable (ω) and the horizontal axis represents the time index (n).

Q9 (a) Write technical note on digital Hilbert transformer and its applications.

Answer

Ans. The frequency response of an ideal Hilbert transformer is given by

$$H(e^{j\omega}) = -j\,\text{sgn}(\omega), \quad |\omega| < \pi$$
$$H(e^{j\omega}) = |\text{sgn}(\omega)|e^{-j\frac{\pi}{2}\,\text{sgn}(\omega)} = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}, \quad |\omega| < \pi$$

or, equivalently

$$H(e^{j\omega}) = -j\,\text{sgn}(\omega) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases} = \begin{cases} e^{-j\frac{\pi}{2}} & 0 < \omega < \pi \\ e^{j\frac{\pi}{2}} & -\pi < \omega < 0 \end{cases}$$

Thus, the magnitude response and phase response of a digital Hilbert transformer are given by

$$|H(e^{j\omega})| = |\text{sgn}(\omega)| = 1 \quad -\pi < \omega < \pi$$
$$\angle H(e^{j\omega}) = -\frac{\pi}{2}\text{sgn}(\omega) = \begin{cases} -\frac{\pi}{2} & 0 < \omega < \pi \\ \frac{\pi}{2} & -\pi < \omega < 0 \end{cases}$$

Note that $H(e^{j\omega})$ is purely imaginary and odd, therefore $h(n)$ will be purely real and odd. Applying the DTFT synthesis equation, we have

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} -j\,\text{sgn}(\omega)e^{j\omega n}d\omega$$

At $n = 0$, we obtain

$$h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -j\,\text{sgn}(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{0} j\,d\omega + \frac{1}{2\pi}\int_{0}^{\pi} -j\,d\omega = \frac{j}{2\pi}[(0+\pi) - (\pi - 0)] = 0$$

Next at $n \neq 0$, we obtain

$$\begin{aligned} h(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} -j\,\text{sgn}(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{0} j\,e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} -j\,e^{j\omega n}d\omega \\ &= \frac{j}{2\pi}\int_{-\pi}^{0} e^{j\omega n}d\omega - \frac{j}{2\pi}\int_{0}^{\pi} e^{j\omega n}d\omega = \frac{j}{2\pi}\left[\frac{e^{j\omega n}}{jn}\Big|_{-\pi}^{0} - \frac{e^{j\omega n}}{jn}\Big|_{0}^{\pi}\right] \\ &= \frac{1}{2n\pi}[1 - e^{-jn\pi} - e^{jn\pi} + 1] = \frac{1}{2n\pi}\left[2 - 2\left(\frac{e^{jn\pi} + e^{-jn\pi}}{2}\right)\right] \\ h(n) &= \frac{1}{2n\pi}[2 - 2\cos(n\pi)] = \frac{1}{n\pi}[1 - \cos(n\pi)] \\ h(n) &= \frac{2}{n\pi}\sin^2\left(\frac{n\pi}{2}\right) \end{aligned}$$

Combining the above results for $n = 0$ and $n \neq 0$, we have the overall result

$$h(n) = \begin{cases} 0 & n = 0 \\ \frac{2}{n\pi}\sin^2\left(\frac{n\pi}{2}\right) & n \neq 0 \end{cases}$$
$$h(n) = \begin{cases} 0 & n = 0,\ n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

which is indeed real-valued and odd, as anticipated.

In practical applications, we seldom require filters that shift the phase for the full frequency range up to $|\omega| = \pi$. If we require phase shifting only up to a cut-off frequency of ω_c, then

$$H(e^{j\omega}) = -j\,\text{sgn}(\omega), \quad |\omega| \leq \omega_c$$

Digital Hilbert transformers find application in modulators and demodulators (single-side band), speech processing, medical imaging, etc. It provides the mathematical basis for the representation of bandpass signals.

Q9 (b) Consider a sequence $x(n)$ with DTFT $X(e^{j\omega})$. The sequence $x(n)$ is real valued and causal and $X_R(e^{j\omega}) = 2 - 2a\cos(\omega)$. Determine $X_I(e^{j\omega})$.

Answer

$$X_R(e^{j\omega}) = 2 - 2a\cos(\omega) = 2 - ae^{j\omega} - ae^{-j\omega}$$

Taking the IDTFT of the above equation and using the fact that $x_e(n) \longleftrightarrow Re\{X(e^{j\omega})\}$, we get

$$x_e(n) = 2\delta(n) - a\delta(n+1) - a\delta(n-1)$$

Since $x(n)$ is causal, we can recover it from $x_e(n)$

$$x(n) = 2x_e(n)u(n) - x_e(0)\delta(n) = 2\delta(n) - 2a\delta(n-1)$$

This implies that

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] = a\delta(n+1) - a\delta(n-1)$$

and since $x_o(n) \longleftrightarrow jX_I(e^{j\omega})$ we find

$$jX_I(e^{j\omega}) = ae^{j\omega} - ae^{-j\omega}$$

$$X_I(e^{j\omega}) = 2a\frac{e^{j\omega} - e^{-j\omega}}{2j} = 2a\sin(\omega)$$

<u>Text Book</u>

Discrete Time Signal Processing (1999), Oppenheim A.V., and Schafer, R.W., with J II, R. Buck, II Edition, Pearson Education, Low Price Edition.