Q2 (a) Determine whether the following signals are periodic or not of periodic then find its fundamental period

(i) $x(n) = (-1)^{n^2}$

(ii) $x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t-2k)$

Answer

(i) $x(n) = (-1)^{n^2}$

For given signal to be periodic

$$x(n+N) = x(n)$$

$$\therefore x(n+N) = (-1)(n+N)^2$$

$$= (-1)^{n^2} + N^2 + 2nN$$

$$= (-1)^{n^2}(-1)^{N^2} + 2nN$$

$$= x(n)(-1)^{N^2} + 2nN \text{.........................(1)}$$

$$\therefore x(n) is \text{ periodic if}$$

$$(-1)^{N^2} + 2nN = 1 = (-1)^{2m}$$

$$for \text{ N} = 1 \quad (-1)^{1+2n} = -1 \neq 1$$

$$for \text{ N} = 2 \quad (-1)^{N^2+2nN} = (-1)^{4+4n} = 1$$

$$\therefore Fundamental \text{ period of x(n) is N} = 2$$

(ii) $x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t-2k)$

For x(t) to be periodic x(t+T)=x(t)

$$\therefore x(t+T) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t+T-2k)$$

Performing a change of variables

$$T - 2T = -2m$$
$$\frac{T}{2} - K = -m$$
$$K = \frac{T}{2} + m$$
$$x(t+T) = \sum_{k=-\infty}^{\infty} (-1)^{\frac{T}{2}+m} \delta(t-2m)$$
$$= \sum_{k=-\infty}^{\infty} (-1)^{m} \delta(t-2m)(-1)^{\frac{T}{2}}$$
$$= x(t)(-1)^{\frac{T}{2}}$$
$$for \text{ x(t) to be periodic}$$
$$(-1)^{\frac{T}{2}} = 1 = (-1)^{2m}$$
$$\frac{T}{2} = 2m$$
$$T = 4m$$
$$\therefore \text{ fundamental period of } x(t) = 4$$

Q2 (b) For each of the following systems determine whether it is Memoryless, Causal, Stable, Linear and Time invariant.

(i) $y(n) = \log_e[x(n)]$

(ii) $y(n) = x(n^2)$

Answer

(i) $y(n) = \log_e[x(n)]$

As the present output depends on the present input only $\therefore$ system is memory less, causal

- Assuming that input signal x(n) satisfies the condition $|x(n)| \le B_x < \infty$ for all n.

We can then final that

$|y(x)| = |\,|\log_e[x(n)]|$

$= |\log_e[Bx| = By < \infty$

$\therefore$ for bounded i/p , system is giving a bounded o/p . Thus system is stable

- consider two arbitrary inputs $|\, x_1(n)$ & $X_2(n)$

$$| x_1(n) \leftrightarrow y_1(n) = \log_e[x_1(n)]$$
$$= x_2(n) \leftrightarrow y_2(n) = \log_e[x_2(n)]$$
$$= let\ \mathrm{x}_3(n) = ax_1(n) + bx_2(n)$$
$$for \text{ system to be linear}$$
$$\mathrm{y}_3(n) = \log_e[x_3(n)] = \log_e[ax_1(n) + bx_2(n)]$$
$$\neq ay_1(n) + by_2(n)$$
$$\therefore System \text{ is Non-Linear}$$
$$\mathrm{y}_1(x) = \log_e[x_1(n)]$$
$$shiftedi/p\ \mathrm{x}_2(n) = x_1(n - n_0)$$
$$O/p \quad \text{corresponding to shifted i/p } \mathrm{y}_2(n) = \log_e[x_2(n)]$$
$$= \log_e[x_1(n - n_0)]$$
$$Now\ \mathrm{y}_1(n - n_0) = \log_e[x_1(n - n_0)]$$
$$\sin ce\ \mathrm{y}_2(n) = \mathrm{y}_1(n - n_0)$$
$$\therefore system \text{ is Time-Invariant}$$

(ii) $y(n) = x(n^2)$

- System has memory since the value of output signal y(x) at time n depends on the future inputs

$y(n)/n = n_0 = y(n_0) = x/n_0^2)$

$\therefore$ It is not memory less

- For bounded i/p system gives a bounded o/p $\therefore$ System is BIBO stable.
- system is Non-Causal since present value of o/p y(n) depends on the future values of input signal x(n)
- Consider two arbitrary inputs $x_1(n) \leftrightarrow y_1(n) = x_1(n^2)$
  $x_2(n) \leftrightarrow y_2(n) = x_2(n^2)$

Let $x_3(n) = ax_1(n) + bx_2(n)$

If system is linear them

$y_3(n) = ay_1(n) + by_2(n)$

$LHS ==> y_3(n) = x_3(n^2) = ax_1(n^2) + bx_2(n^2)$

$= ay_{1(n)} + by_2(n) = RHS$

$Since \quad LHS = RHS \; \therefore$ System is linear

$* \, \mathrm{y}_1(n) = x_1(n^2) = ax_1(n^2) + bx_2(n^2)$

$= \mathrm{ay}_1(n) + \mathrm{by}_2(n) = RHS$

$Since \quad \mathrm{LHS} = \mathrm{RHS} \; \therefore$ system is linear

$* \, \mathrm{y}_1(n) = x_1(n^2)$

consider the shifted i/p $\mathrm{x}_2(n) = x_1(n - n_0)$ o/p corresponding to shifted i/p $\mathrm{y}_2(n) = x_2(n^2)$

$\mathrm{y}_2(n) = x_1(n^2 - n_0)$

$Now \;\; \mathrm{y}_1(n - n_0) \; = \mathrm{x}_1((n - n_0)^2)$

$\therefore y_2(n) \neq y_1(n - n_0)$

$\therefore$ *system* is Time - Variant

Q3 (a) Find the trigonometric Fourier series for the triangular wave shown in Fig.1 and hence plot its line spectrum.

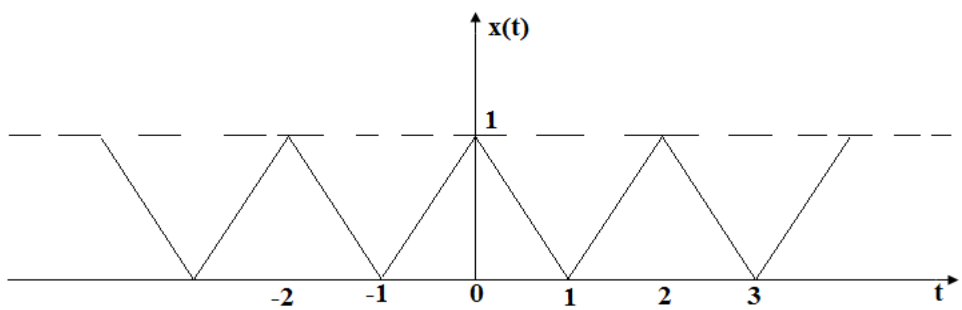


Fig.1

Answer

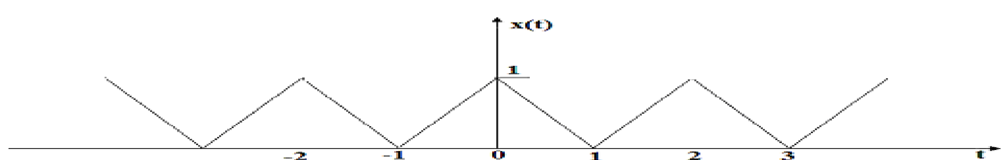


Fig.1

Waveform is periodic write period T=2

& Fundamental Frequency $w_0 = \frac{2\pi}{T}$

$$x(t) = \begin{cases} 1-t & 0 < t < 1 \\ t-1 & 0 < t < 2 \end{cases}$$

The Wave is an even function $\therefore b_n = 0$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)dt = \frac{2}{2}\int_0^1 (1-t)dt$$

$$=> \left| t - \frac{t^2}{2} \right|_0^1 = \frac{1}{2}$$

$$= a_n = 4/T \int_0^{T/2} x(t)\cos(nw_0 t)dt$$

$$= \frac{4}{2}\int_0^1 (1-t)\cos(n\pi t)dt$$

$$= 2\left[(1-t)\sin\frac{n\pi t}{n\pi} \Big)_0^1 - \left| \frac{\cos n\pi t}{n^2\pi^2} \right|_0^1 \right]$$

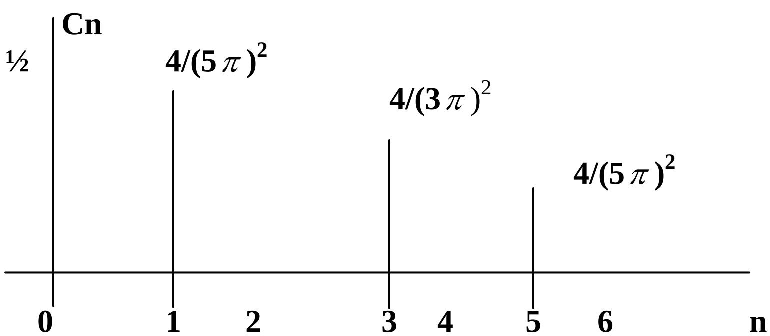
$$= \frac{2}{n^2\pi^2}\left[1 - \cos(n\pi)\right]$$

$$a_n = \begin{cases} \frac{4}{n^2\pi^2} & n = 1,3,5.......... \\ 0 & n = 2,4,6................. \end{cases}$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_0 t)$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\cos(\pi t) + \frac{4}{(3\pi)^2}\cos(3\pi t) + \frac{4}{(5\pi)^2}\cos(5\pi t)...... +$$

line spectrum $c_n = \sqrt{an^2 + bn^2} = |a_n|$



Q3 (b) A continuous time periodic signal is real valued and has a fundamental period T= 8 .The non zero Fourier series coefficients for x(t) are $X_1 = X_{-1} = 2$, $X_3 = X^*_{-3} = 4j$. **Express x(t) in the form** $x(t) = \sum_{n=0}^{\infty} A_n \text{Cos}(\omega_n t + \Phi_n)$

Answer

Given:

$$T = 8 \therefore w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$we \text{ have } x(t) = \sum_{n=-\infty}^{\infty} X_n e^{-jw_0 t} + X_{-3} e^{-j3w_0 t}$$

$$= 2e^{j\frac{\pi}{4}t} + 2e^{-j\frac{\pi}{4}t} + 4je^{j3\pi/4t} - 4je^{-j3\pi/4t}$$

$$= 4\cos(\frac{\pi}{4}t) - 8\sin\left(\frac{3\pi}{4}t\right)$$

$$x(t) = 4\cos(\frac{\pi}{4}t) + 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2})$$

Q3 (c) Find the time domain signal corresponding to following DTFS coefficients

Answer

$$X_k = \cos\left(\frac{k4\pi}{11}\right) + 2j\sin\left(\frac{k6\pi}{11}\right)$$

$$= \frac{1}{2}e^{j4\frac{\pi}{11}k} + \frac{1}{2}e^{-j4\frac{\pi}{11}k} + e^{j6\frac{\pi}{11}k} - e^{-j6\frac{\pi}{11}k}$$

$$= \frac{1}{2}e^{j2\frac{\pi}{11}2k} + \frac{1}{2}e^{-j2\frac{\pi}{11}2k} e^{j2\frac{\pi}{11}.11k} + e^{j2\frac{\pi}{11}.3k} - e^{j2\frac{\pi}{11}11k}$$

$$= \frac{1}{2}e^{j2\frac{\pi}{11}2k} + \frac{1}{2}e^{j2\frac{\pi}{11}9k} + e^{j2\frac{\pi}{11}3k} - e^{j2\frac{\pi}{11}8k}$$

$$= \frac{1}{11}\sum_{k=0}^{10}\left[\frac{11}{2}\delta(n-2) + 11\delta(n-3) - 11\delta(n-8) + \frac{11}{2}\delta(n-1)\right]e^{j2\frac{\pi}{11}nk}$$

$$X_k = \frac{11}{N}\sum_{K=0}^{N-1} x(n)e^{j2\frac{\pi}{N}nk,}$$

Comparing the above two equations

$$= x(n) = \frac{11}{2}\delta(n-2) + 11(n-3) - 11\delta(n-8) + \frac{11}{2}\delta(n-9)$$

Q4 (a) State and Prove duality property of Continuous Time Fourier Transform. Using it, find the Fourier Transform of following signals

(i) $g(t) = \frac{1}{1+jt}$

(ii) $x(t) = \frac{1}{1+t^2}$

Answer

Duality property of $x(t) \leftrightarrow x(w)$

then $x(t) \leftrightarrow 2\pi x(-w)$

proof :- $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(w)e^{jwt}\,dw$$

$$2\pi x(t) = \int_{-\infty}^{\infty} x(w)e^{jwt}\,dw$$

Replace t by - t

$$2\pi x(-t) = \int_{-\infty}^{\infty} x(w)e^{-jwt}\,dw$$

Interchanging the variables t and w.

$$2\pi x(-w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

$$2\pi x(-w) = F[x(t)]$$

$$x(t) \leftrightarrow 2\pi x(-w)$$

(i) $g(t) = \frac{1}{1+jt}$

Define $x(w) = \frac{1}{1+jw}$ & *replace* t with w in the expression of g(t)

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+jw}$$

Subsitituting $a = 1$

$$a^{-t}u(t) \leftrightarrow \frac{1}{1+jw}$$

$$x(t) = a^{-t}u(t) \quad \& \quad x(w) = \frac{1}{1+jw}$$

$$x(-w) = a^{w}u(-w) \quad \& \quad x(t) = \frac{1}{1+jt}$$

According to duality

$$x(t) \leftrightarrow 2\pi x(-w)$$

$$\frac{1}{1+jt} \leftrightarrow 2\pi e^{w}u(-w)$$

$$f\left[\frac{1}{1+jt}\right] = 2\pi e^{w}u(-w)$$

$(ii)\quad x(t) = \dfrac{1}{1+t^2}$

we have, $e^{-a|t|} \leftrightarrow \dfrac{2a}{a^2+w^2}$

Put $\quad a = 1 \quad e^{-a|t|} \leftrightarrow \dfrac{2}{1+w^2}$

$$\frac{1}{2}e^{-a|t|} \leftrightarrow \frac{1}{1+w^2}$$

$x(t) = \dfrac{1}{2}e^{-a|t|}$ and $x(w) = \dfrac{1}{1+w^2}$

$x(-w) = \dfrac{1}{2}e^{-|-w|} \qquad x(t) = \dfrac{1}{1+t^2}$

$= \dfrac{1}{2}e^{-|-w|}$

Acc.to Duality

$X(t) \leftrightarrow 2\pi x(-w)$

$= \dfrac{1}{1+t^2} \leftrightarrow 2\pi.\dfrac{1}{2}e^{-|w|}$

$= \dfrac{1}{1+t^2} \leftrightarrow \pi.e^{-|w|}$

$= F\left[\dfrac{1}{1+t^2}\right] = \pi.e^{-|w|}$

Q4 (b) Consider a stable LTI system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

(i) Find the frequency response $H(\omega)$ **and impulse response h(t) of the system.**

(ii) What is the response of this system if the input $x(t) = e^{-t}u(t)$

Answer

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Taking Fourier Transform on both sides

$$(jw)^2 y(w) + 4jwy(w) + 3y(w) = jwx(w) + 2x(w)$$

$$H(w) = \frac{y(w)}{x(w)} = \frac{2+jw}{(jw)^2 + 4jw + 3} = \frac{2+jw}{(1+jw)(3+jw)}$$

$$H(w) = \frac{1}{2}.\frac{1}{1+jw} + \frac{1}{2}.\frac{1}{3+jw}$$

Jaking inverse fouries Transform

$$\text{h(t)} = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

(*ii*) Given $\text{x(t)} = \text{e}^{-t}u(t)$

$$\text{x(w)} = \frac{1}{1+\text{jw}}$$

$$y(w) = \text{x(w).H(w)} = \left[\frac{1}{1+\text{jw}}\right]\left[\frac{2+jw}{(1+jw)(3+jw)}\right]$$

$$==> \frac{2+jw}{(1+jw)^2(3+jw)}$$

$$y(w) = \frac{1}{4}\frac{1}{1+jw} + \frac{1}{2}\frac{1}{(1+jw)^2} - \frac{1}{4}\frac{1}{(3+jw)}$$

Taking inverse fouries Transform

$$\text{y(t)} = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$

Q5 (a) Suppose that a system has the response $(\frac{1}{4})^n u(n)$ **to the input** $(n+2)(\frac{1}{2})^n u(n)$ **.If the output of this system is** $\delta(n) - (-\frac{1}{2})^n u(n)$ **,what is the input?**

Answer

$$Given\ \text{that x(n)} = (\text{n}+3)\left(\frac{1}{2}\right)^n u(n)$$

$$= n\left(\frac{1}{2}\right)^n 4(n) + 2\left(\frac{1}{2}\right)^n 4(n)$$

$Taking$ DTFT of the above equation

$$\text{x}(\text{e}^{\text{jw}}) = \frac{\frac{1}{2}e^{-jw}}{\left(1-\frac{1}{2}e^{-jw}\right)^2} + 2\frac{1}{1-\frac{1}{2}e^{-jw}}$$

$$= \frac{2\left(1-\frac{1}{4}e^{-jw}\right)}{\left(1-\frac{1}{2}e^{-jw}\right)^2}$$

$$Given\ \text{y(n)} = \left(\frac{1}{4}\right)^n u(n)$$

$$y(e^{jw}) = \frac{1}{1-\frac{1}{4}e^{-jw}}$$

$$H(e^{jw}) = \frac{y(e^{jw})}{x(e^{jw})} = \frac{\left(1-\frac{1}{2}e^{-jw}\right)^2}{2\left(1-\frac{1}{4}e^{-jw}\right)}$$

$$Given\ \text{y(x)} = \delta(\text{n}) - \left(-\frac{1}{2}\right)^n u(n)$$

$$y(e^{jw}) = 1 - \frac{1}{1+\frac{1}{2}e^{-jw}} = \frac{\frac{1}{2}e^{-jw}}{1+\frac{1}{2}e^{-jw}}$$

$$x(e^{jw}) = \frac{2\left(1-\frac{1}{4}e^{-jw}\right)^2}{\left(1-\frac{1}{2}e^{-jw}\right)^2} = y(e^{jw})$$

$$= \frac{2\left(1-\frac{1}{4}e^{-jw}\right)}{\left(1-\frac{1}{2}e^{-jw}\right)^2} = \frac{\frac{1}{2}e^{-jw}}{1+\frac{1}{2}e^{-jw}}$$

$$= \frac{e^{-jw}\left(1-\frac{1}{4}e^{-jw}\right)^2}{\left(1-\frac{1}{2}e^{-jw}\right)^2\left(1+\frac{1}{2}e^{-jw}\right)}$$

$$= \frac{\frac{3}{8}e^{-jw}}{1+\frac{1}{2}e^{-jw}} + \frac{\frac{3}{8}e^{-jw}}{1-\frac{1}{2}e^{-jw}} + \frac{\frac{1}{8}e^{-jw}}{\left(1-\frac{1}{2}e^{-jw}\right)^2}$$

Taking inverse DTFT

$$\mathrm{x(n)} = \frac{3}{8}\left(-\frac{1}{2}\right)^{n-1} u(n-1) + \frac{3}{8}\left(\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{4}n\left(\frac{1}{2}\right)^{n} u(n)$$

Q5 (b) State and Prove convolution property of Discrete Time Fourier Transform. Using it determine the convolution $x(n) = x_1(n) * x_2(n)$ **of the sequences, where**

$x_1(n) = x_2(n) = \delta(n+1) + \delta(n) + \delta(n-1)$

Answer

Convolution property:

If

$x_1(n) \leftrightarrow x_1(e^{jw}) \qquad x_2(n) \leftrightarrow x_2(e^{jw})$

then $x_1(n) + x_2(n) \leftrightarrow x_1(e^{jw})\, x_2(e^{jw})$

proof :- $f[x_1(n) + x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) + x_2(n)] e^{-jwn}$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m) \right) e^{-jwn}$$

Interchanging the order of summation

$$= \sum_{m=-\infty}^{\infty} x_1(m) \left(\sum_{n=-\infty}^{\infty} x_2(n-m) e^{-jwn} \right)$$

Apply time shifting property to the bracketed term

$$= x_2(e^{jw}) x_1(e^{jw}) = x_1(e^{jw}) x_2(e^{jw})$$

$$= x_1(n) \times x_2(n) \leftrightarrow x_1(e^{jw}) x_2(e^{jw})$$

$$x_1(n) = x_2(n) = \{"\uparrow"\} = \delta(n+1) + \delta(n) + \delta(n-1)$$

$$x_1(e^{jw}) = x_2(e^{jw}) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-jwn}$$

$$= \sum_{n=-1}^{1} e^{-jwn}$$

$$= e^{jwn} + 1 + e^{-jwn}$$

$$= 1 + 2\cos w$$

Using condition property,

$$F[x_1(n) + x_2(n)] = x_1(e^{jw}).x_2(e^{jw})$$

$$= [1 + 2\cos w]^2$$

$$= 3 + 4\cos w + 2\cos(2w)$$

$$= 3 + 2(e^{jw} + e^{-jw}) + (e^{j2w} + e^{-j2w})$$

$$= e^{j2w} + 2e^{jw} + 3 + 2e^{-jw} + e^{-j2w}$$

$$= x(n) = \delta(n+2) + 2\delta(n+1) + 3\delta(n) + 2\delta(n-1) + \delta(n-2)$$

$$= \{1,2,3,,2,1\}$$

Q6 (a) Determine the conditions on the sampling interval T_S so that each x(t) is uniquely represented by the discrete time sequence $x(n) = x(nT_S)$.

(i) $x(t) = \cos(\pi t) + 3\sin(2\pi t) + \sin(4\pi t)$

(ii) $x(t) = \cos(2\pi t)\sin c(t) + 3\sin(6\pi t)\sin c(2t)$

Answer

$x(t) = \cos(\pi t) + 3\sin(2\pi t) + \sin(4\pi t)$

Comparing it with

$$x(t) = A_1\cos(w_1 t) + A_2 \sin(w_2 t) + A_3 \sin(w_3 t)$$

$$w_1 = \pi, w_2 = 2\pi, w_3 = 4\pi$$

$$W_{mox} = w_3 = 4\pi \quad \text{or } f_{mox} = \frac{w_{max}}{2\pi} = 2$$

$$\therefore \text{Sampling frequency } w_s \geq 2w_{max} = 8\pi ras/\sec$$

$$or \quad f_s \geq 2f\max = 4Hz$$

$$\textit{Hence} \text{ sampling interval} \quad T_s \leq \frac{1}{4}$$

(*ii*) Given $x(t) = \cos(2\pi t)\sin c(t) + 3\sin(6\pi t)\sin c(2t)$

$$= \cos(2\pi t)\frac{\sin(\pi - t)}{\pi t} + 3\sin(6\pi t)\frac{\sin(2\pi t)}{2\pi t}$$

$$= \frac{1}{(2\pi t)}\left[\sin(3\pi t) - \sin(\pi t)\right] + 3\sin(6\pi t)\frac{\sin(2\pi t)}{2\pi t}$$

$$= \frac{1}{2\pi t}\left[\sin(3\pi t) - \sin(\pi t)\right] + \frac{3}{4\pi t}\left[\cos(4\pi t) - \cos(8\pi t)\right]$$

$$\text{Maximum freq. is } w_{mox} = 8\pi \quad \text{or } f_{max} = \frac{w_{mox}}{2\pi} = 4$$

$$\text{Sampling freq. } f_s \geq 2f_{max} = 8Hz$$

$$\text{Sampling Interval} \quad T_s \leq \frac{1}{8}$$

Q6 (b) A causal LTI system is described by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Determine:

(i) The frequency response of the system

(ii) The group delay associated with the system

(iii) Output of the system to the input $x(t)=e^{-t}u(t)$

(iv) Output of the system if the input has its fourier transform

$$X(jw) = \frac{j\omega + 1}{(j\omega + 2)}$$

Answer

(i) Taking F.transform both the sides of given equ

$$jw \quad y(w) + 2y(jw) = x(jw)$$
$$H(jw) = \frac{y(jw)}{x(jw)} = \frac{1}{jw+2}$$

$$(ii) \quad \angle H(jw) = -\tan^{-1}\left(\frac{w}{2}\right)$$
$$T(w) = -\frac{d}{dw}\{\angle H(jw)\} = -\frac{d}{dw}\left\{-\tan^{-1}\left(\frac{w}{2}\right)\right\}$$
$$= \frac{1}{1+\left(\frac{w}{2}\right)^2} \cdot \frac{1}{2} = \frac{2}{4+w^2}$$

$$(iii) \quad x(t) = e^{-t}u(t)$$
$$x(jw) = \frac{1}{jw+1}$$
$$y(jw) = H(jw) \times (jw) = \frac{1}{(jw+1)(jw+2)}$$
$$= \frac{1}{(jw+1)} - \frac{1}{(jw+2)}$$
Taking inverse fouries transform
$$y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

$$(iv) H(jw).X(jw) = \frac{jw+1}{(jw+2)^2}$$
Taking inverse fouries transform
$$y(t) = \left[e^{-2t} - te^{-2t}\right]4(t)$$

Q7 (a) Consider the signal $x(t) = e^{-5t}u(t-1)$ **and its Laplace Transform be X(s)**

- **(i) Evaluate X(s) and find its ROC**
- **(ii) Determine the values of the finite numbers A and t_0 such that the Laplace transform G(s) of** $g(t) = Ae^{-5t}u(-t-t_0)$ **has the same algebraic form as X(s).What is the ROC corresponding to G(s)?**

Answer

(i) By definition

$$x(5) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(s) = \int_{-\infty}^{\infty} e^{-st}u(t-1)e^{-st}dt$$

$$u(t-1) = \begin{cases} 1 & t>1 \\ 0 & t<1 \end{cases}$$

$$x_{(s)} = \int_{1}^{\infty} e^{-st}e^{-st}dt = \int_{1}^{\infty} e^{-1(5+5)t}dt$$

$$= \left| \frac{e^{-(5+5)t}}{-(5+5)} \right|_{1}^{\infty} = -(s+5)\left[0 - e^{-(5+5)t}\right]$$

$$= \frac{e^{-(5+5)t}}{s+5} \quad \text{R.O.C R\{s\} > -5}$$

(ii)

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} Ae^{-5t}u(-t-t_0)e^{-st}dt$$

$$u(-t-t_0) = \begin{cases} 1(-t-t0) > 0 \rightarrow t < -t0 \\ 0(-t-t0) < 0 \rightarrow t > -t0 \end{cases}$$

$$G(s) = \int_{-\infty}^{-t_0} Ae^{-5t}e^{-st}dt = \int_{-\infty}^{-t0} Ae^{-(s+5)}dt$$

$$= A\left| \frac{-Ae^{(s+5)t_0}}{s+5} \right|_{-\infty}^{-t0}$$

$$G(s) = \frac{-Ae^{(s+5)t_0}}{s+5} \text{.........} G(8) = x(8) if$$

$$\text{A} = \text{-1, to} = \text{-1.}$$

R.O.C $\qquad$ $\text{R}\{\text{s}\} < -5$

Q7 (b) Find the inverse Laplace transform of $X(s) = \dfrac{-3}{(s+2)(s-1)}$

If the ROC is:

(i) Re{s} > 1
(ii) Re{s} < - 2
(iii) -2 < Re{s} < 1

Answer

$$X(s) = \frac{-3}{(s+2)(s-1)}$$

$$= \frac{1}{(s+1)} - \frac{1}{(s-1)}$$

$$X(s) has poles at -2 and 1$$

(i)

$ROC R\{s\} > 1$

Is to the right of the eight most poles so both poles correspond to casual signals.

$\therefore$

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}$$

$$e^{t}u(t) \leftrightarrow \frac{1}{s-1}$$

and

$$x(t) = e^{-2t}u(t) - e^{t}u(t)$$

(ii)

$R(s) < -2$ is to the left of leftmost pole so both poles correspond to anticasual signals $\therefore$

$$-e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}$$

$$-e^{-t}u(t) \leftrightarrow \frac{1}{s-1}$$

and

$$x(t) = -e^{-2t}u(-t) + e^{t}u(t)$$

$(iii) -2 < R\{s\} < 1$

for pole at –2 ROC lies to the right of this pole .:. This pole corresponds to a casual signal.

$$\therefore\ e^{-2t}u(t) \leftrightarrow \frac{1}{s+1}$$

Second pole is at s=1. Hence ROC is to the left of this pole. So this pole corresponds to the ant casual

$$\therefore -e^{t}u(-t) \leftrightarrow \frac{1}{s-1}$$

Hence

$$x(t) = e^{-2t}4(t) + e^{t}u(-t)$$

Q8 (a) Determine the signal x(n) whose z-transform is given by $X(z) = \log(1 + az^{-1}), |z| > |a|$

Answer

$$X(z) = \log(1 + az^{-1}), |z| > |a|$$

$$\frac{dx(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$$

$$-Z(\frac{dx(z)}{dz}) = \frac{-az-1}{1+az-1}$$

Take inverse Z-transform

$$(z^{-1}) = \left[- = \frac{dx(z)}{dz} \right] = z^{-1}\left[\frac{-az-1}{1+az-1} \right]$$

$$nx(n) = z^{-1}\left[\frac{-az^{-1}}{1+az^{-1}} \right]$$

we know

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}}$$

$$(-a)^n u(n) \leftrightarrow \frac{1}{1+az^{-1}}$$

$$|Z| > |a|$$

$$a(-a)^n u(n) \leftrightarrow \frac{a}{1+az^{-1}} \; |z| > |a|$$

Using time –shifting property,

$$a(-a)^{n-1} u(n-1) \leftrightarrow \frac{az^{-1}}{1+az^{-1}} \; |z| > |a|$$

$$-(-a)^n u(n-1) \leftrightarrow \frac{az^{-1}}{1+az^{-1}} \; |z| > |a|$$

consequently

$$nx(n) = -(-a)^n u(n-1)$$

$$x(n) = \frac{-(-a)^n}{n} u(n-1) = \frac{1}{n}(-1)^{n+1} a^n u(n-1)$$

Q8 (b) Find the inverse z-transform of $X(z) = \dfrac{1+z^{-1}}{1-(1/3)z^{-1}}$

When,

(i) ROC: |z| > 1/3

(ii) ROC : |z| < 1/3, using power series expansion

Answer

$$1+\frac{4}{3}z^{-1}+\frac{4}{9}z^{2}+\frac{4}{27}z^{-3}+......$$

$$1-1/3z^{-1}\,\big)\,1+z^{-1}$$

$$1-\frac{1}{3z^{-1}}$$

$$\frac{4}{3}z^{-1}-\frac{4}{9}z^{-2}$$

$$\frac{4}{9}z^{-2}$$

$$\frac{4}{9}z^{-2}-\frac{4}{27}z^{-3}$$

$$\frac{4}{27}z^{-3}$$

$$\therefore x(z)=1+\frac{4}{3}z^{-1}+\frac{4}{9}z^{-2}+\frac{4}{27}z^{-3}......$$

$$\therefore x(n)=\left\{0,\frac{4}{3},\frac{4}{9},\frac{4}{27},.............\right\}$$

(ii)

$$-3-12z-36z^{2}+.....$$

$$-\frac{1}{3}z^{-1}+1\,\big)\,z^{-1}+1$$

$$z^{-1}-3$$

$$4$$

$$4-12z$$

$$12z$$

$$12z-36z^{2}$$

$$36z^{2}$$

Q9 (a) A random variable X has the uniform distribution given by

$$f_X(x) = \begin{cases} \frac{1}{2\pi}, & \text{for } 0 \le x \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Determine its mean and variance

Answer

$$m_x = \int_{-\infty}^{\infty} xfx(x)dx = \int_0^{2\pi} x.\frac{1}{2\pi}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.dx = \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\rho} = \pi$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 fx(x)dx = \int_0^{2\pi} x^2.\frac{1}{2\pi}dx$$

$$= 1/2\pi \int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} = \frac{4}{3}\pi^2$$

$$\text{var}\,iance x^2 = E[x^2] - \left[E[x]\right]^2$$

$$= \frac{4}{3}\pi^2 - \pi^2 = \frac{\pi^2}{3}$$

Q9 (b) A WSS random process X(t) with autocorrelation function $R_X(\tau) = e^{-a|\tau|}$ where a is a real positive constant is applied to the input of LTI system with impulse response h(t) = e^{-bt} u(t) where b is real positive constant. Find the autocorrelation function of the output Y(t) of the system.

Answer

Frequency response H/W of the system

H (W) =F [h(t)]=1/jw+b

Power spectral density of X(t) is

Sx (w) = F [Rx(z)]=

$$\frac{2}{w^2 + a^2}$$

$$sr(w) = |H(w)|^2\ sx(w) = \left(\frac{1}{w^2+b^2}\right)\left(\frac{2a}{w^2+b^2}\right)$$

$$= \frac{a}{(a^2-b^2)b}\left(\frac{2b}{w^2+b^2}\right) - \frac{b}{(a^2-b^2)b}\left(\frac{2a}{w^2+a^2}\right)$$

Taking inverse Fourier transform on both sides

$$R\lambda(c) = \frac{1}{(a^2+b^2)b}\left[ae^{-b/e|} - be^{-a|e|}\right]$$

Text Books

1. Signals and Systems, A.V. Oppenheim and A.S. Willsky with S. H. Nawab, Second Edition, PHI Private limited, 2006.

2. Communication Systems, Simon Haykin, 4th Edition, Wiley Student Edition, 7th Reprint 2007.