

Q2 (a) If $x^x y^y z^z = c$, show that at $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Answer

Log of $x^x y^y z^z = c$, we get

$$x \log x + y \log y + z \log z = \log c \dots \dots \dots (1)$$

Diff both sides w.r.t x and y

$$x \frac{1}{x} + \log x + z \frac{1}{2} \frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \dots \dots \dots (2)$$

$$y \frac{1}{y} + \log y + z \frac{1}{2} \frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0 \text{ or } \frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z} \dots \dots \dots (3)$$

Diff (3) paste all w.r.t. x , we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1 + \log y}{(1 + \log z)^2} \cdot \frac{1}{z} \frac{\partial z}{\partial x}$$

$$= -\frac{1 + \log y}{(1 + \log z)^2} \frac{1}{z} \frac{1 + \log x}{1 + \log z} \text{ (using (2))}$$

$$\text{At } x = y = z$$

$$= -\frac{1 + \log x}{(1 + \log x)^2} \frac{1}{x} \frac{1 + \log x}{1 + \log x} = -\frac{1}{x(1 + \log x)} = -[x(\log e + \log x)]^{-1}$$

$$= -[x \log ex]^{-1}$$

Q2 (b) Expand $f(x, y) = \tan^{-1}(xy)$ in powers of $(x - 1)$ and $(y - 1)$ upto second degree terms.

Answer

Here,

$$f(x, y) = \tan^{-1}(xy) \therefore f(1,1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f_x(x, y) = \frac{y}{1+x^2y^2} \therefore f_x(1,1) = \frac{1}{2}$$

$$f_y(x, y) = \frac{x}{1+x^2y^2} \therefore f_y(1,1) = \frac{1}{2}$$

$$f_{xx}(x, y) = -\frac{2y^3x}{(1+x^2y^2)^2} \therefore f_{xx}(1,1) = -\frac{1}{2}$$

$$f_{yy}(x, y) = -\frac{2x^3y}{(1+x^2y^2)^2} \therefore f_{yy}(1,1) = -\frac{1}{2}$$

$$f_{xy}(x, y) = -\frac{1}{1+x^2y^2} - \frac{2x^2y^2}{(1+x^2y^2)^2} \therefore f_{xy}(1,1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\therefore f(x, y) = \tan^{-1}(xy) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) + \frac{1}{2}(x-1)^2 f_{xx}(1,1) + \frac{1}{2}(y-1)^2$$

$$f_{yy}(1,1) + (x-1)(y-1)f_{xy}(1,1) + \dots$$

$$= \frac{\pi}{4} + (x-1)\frac{1}{2} + (y-1)\frac{1}{2} + \frac{1}{2}(x-1)^2\left(-\frac{1}{2}\right) + \frac{1}{2}(y-1)^2\left(-\frac{1}{2}\right) + (x-1)(y-1)\dots$$

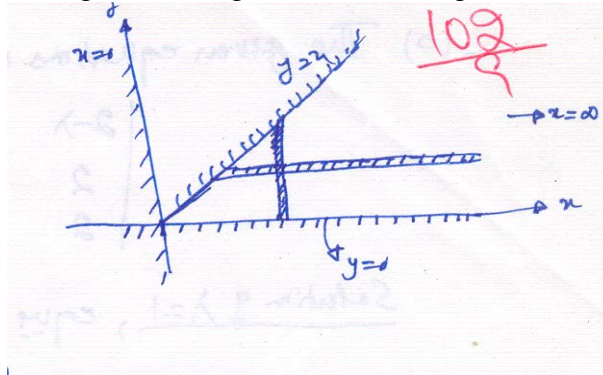
$$= \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \dots$$

Q3 (a) Change the order of integration and then evaluate it $\int_0^\infty \int_0^x xe^{-x/y} dy dx$

Answer

To change order of integration, we take strip parallel to x-axis to cover the each bounded by $x = 0, x = \infty, y = 0, y = x$.

Strip moves parallel to itself from $y=0$ to $y=\infty$, keeping its ends on $x=y$ two $x=\infty$ hence the integral in changed order in changed order is



$$\int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y} dx dy$$

$$= \int_0^{\infty} \left[-\frac{y}{2} e^{-x^2/y} \right]_y^{\infty} dy$$

$$= \int_0^{\infty} -\frac{y}{2} (0 - e^{-y}) dy = \frac{1}{2} \int_0^{\infty} y e^{-y} dy = \frac{1}{2} \left[-y e^{-y} \Big|_0^{\infty} + \int_0^{\infty} e^{-y} dy \right]$$

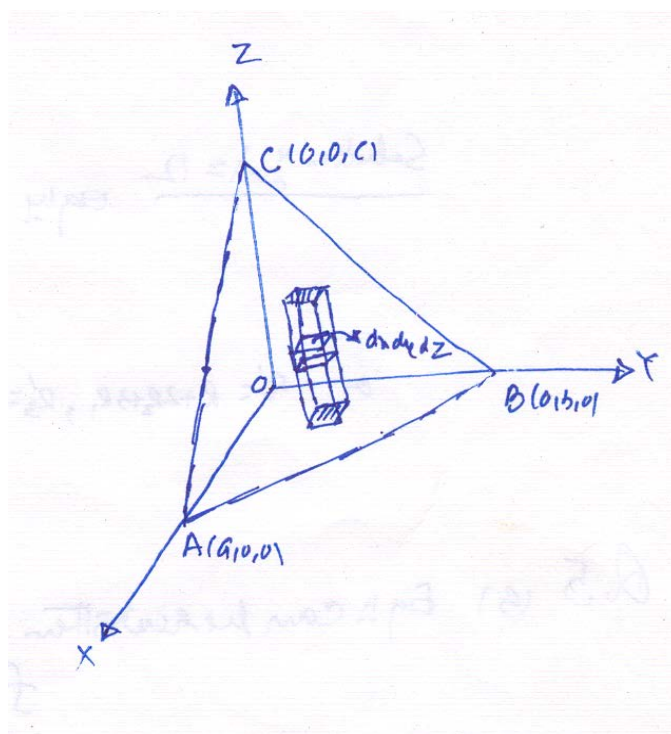
$$= \frac{1}{2} \left[-e^{-y} \right]_0^{\infty} = \frac{1}{2}$$

Q3 (b) Find the volume of the tetrahedron bounded by the coordinate planes and

$$\text{the plane } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Answer

Volume of the tetrahedron bounded by the coordinate to plnes and the plain



$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1n$$

$\iiint dz dy dx$ taken one entire val

$$\int_0^{a b^{(1-n)} z} \int_0^a \int_0^{b(1+y)} dz dy dx = \int_0^a \int_0^a z dy du$$

$$= \int_0^{a b^{(1-n/a)}} \int_0^a c \left(1 - \frac{x}{a} - \frac{y}{b}\right) dy dx$$

$$= c \int_0^a \left[y - \frac{x}{a} y - \frac{1}{b} \frac{y^2}{2} \right]_0^{b(1-n/a)} dx = c \int_0^a \left[b \left(1 - \frac{n}{a}\right) - \frac{nb}{a} \left(1 - \frac{n}{a}\right) - \frac{b^2}{2b} \left(1 - \frac{n}{a}\right)^2 \right] dx$$

$$= c \left[b \left(a - \frac{a^2}{2a}\right) - \frac{b}{a} \left(\frac{a^2}{2} - \frac{1}{a} \frac{a^3}{3}\right) - \frac{b}{2} \left(a - \frac{2}{a} \frac{a^2}{2} + \frac{1}{a^2} \frac{a^3}{3}\right) \right]$$

$$= c \left[ab - \frac{ab}{2} - \frac{ab}{2} + \frac{ab}{3} - \frac{ab}{2} + \frac{ab}{2} - \frac{ab}{6} \right] = \frac{abc}{6}$$

Q4 (a) Solve the equation
$$\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Answer

$$0 = \begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = \begin{vmatrix} x+1 & x & x \\ 2x & 2x+3 & 2x \\ 4x+1 & 2x+3 & 2x \end{vmatrix} = \begin{vmatrix} x+1 & x & x \\ -2 & 3 & 0 \\ 2x+1 & 0 & 0 \end{vmatrix} = -3x(2x+1)$$

Hence $x = 0$ or $x = -\frac{1}{2}$

Q4 (b) Find the values of λ for which the equations $(2-\lambda)x + 2y + 3 = 0$, $2x + (4-\lambda)y + 7 = 0$, $2x + 5y + (6-\lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ .

Answer

The given equation will be consistent if

$$\begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{vmatrix} = 0 \text{ i.e}$$

$$\lambda^3 - 12\lambda^2 - \lambda + 12 = 0$$

i.e

$$\lambda = -1, = +1 = 12$$

solving $\lambda = 1$, equation become

$$x + 2y + 3 = 0$$

$$2x + 3y + 7 = 0$$

$$2x + 5y + 5 = 0$$

3rd equation is 4 times (1)-(2). \therefore 3rd n independent solution given by (1) and (2) and it is $x=-5, y=1$

Solving $\lambda = -1$, equation become

$$\begin{bmatrix} 3 & 2 \\ 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ +6 \end{bmatrix}$$

or

$$R_1^1 = R_1 + 2R_2 + 3R_3, R_3^1 = R_3 - R_2 \begin{bmatrix} 0 & 1 \\ 2 & -8 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 13 \end{bmatrix}$$

Hence $y = 1, x = \frac{1}{2}$

Q5 (a) Use Regula-Falsi method to compute the real root of $xe^x = 2$ correct to three decimal places.

Answer

Equation can be written as

$$f(x) = xe^x - 2 = 0 \dots\dots\dots(1)$$

$$f(0) = -2 \text{ and } f(1) = 0.718281828$$

\therefore Root of (1) line between 0 and 1. By Regula falsi method , first approach .is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-1(-2)}{2.718281828} = 0.73575888$$

$$f(x_1) = f(0.73575888) = -0.464423228$$

\therefore Root line between 0.73575888 and 1.su second approach .root is given by

$$x_2 = \frac{0.73585888(.718281828) - 1(-.464423228)}{.718281828 + .464423228} = +0.83952077$$

$$f(x_2) = f(+.83952077) = -0.0562935$$

\therefore root line between 0.82952077 and 1. so third approach . root is given by

Q5 (b) Use Runge-Kutta method of order four to find y(0.2) for the equation

$$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1. \text{ Take } h = 0.2.$$

Answer

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}, x_0 = 0, y_0 = 1, h = 0.2$$

$$\therefore k_1 = hf(x_0, y_0) = 0.2x \frac{1-0}{1+0} = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{ky}{2}\right) = 0.2 \frac{(1+.1) - (0+.1)}{(1+.1) + (0+.1)} = \frac{0.2}{1.2} = 0.16666$$

$$\text{Here } k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{ky}{2}\right) = 0.2 \frac{(1+.08333) - (0+.1)}{(1+.08333) + (0+.1)} = 0.16619$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \frac{(1+.16619) - (0+.2)}{(1+.16619) + (0+.2)} = 0.14144$$

$$\therefore k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 0.167836$$

$$\text{Hence } y(0.2) = 1.167836$$

Q6 (a) Solve the equation $\frac{dy}{dx} = -\left(\frac{x + y \cos x}{1 + \sin x}\right)$

Answer

Given equation can be written as

$$(x+y \cos x)dx + (1+\sin x)dy = 0$$

Here $M = x + y \cos x$, $N = 1 + \sin x$

$$\frac{\partial M}{\partial y} = \cos x = \frac{\partial N}{\partial x}$$

\therefore Given equ in each hence required solution is

$$\frac{x^2}{2} + y \sin x + y = k (\text{a constant})$$

Q6 (b) Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda y + C = 0$, λ being the parameter.

Answer

Given family of coaxial circles is

$$x^2 + y^2 + 2\lambda y + C = 0 \dots \dots \dots (1)$$

diff (1), we get

$$2x + 2y \frac{dy}{dx} + 2\lambda \frac{dy}{dx} = 0 \dots \dots \dots (2)$$

\therefore Wronskian is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

$$\begin{aligned} \therefore \text{P.I} &= -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx \\ &= -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2} dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \frac{e^{3x}}{x^2} dx \\ &= -e^{3x} \log x - e^{3x} \end{aligned}$$

Hence compute solution is

$$y = (c_1 + c_2 x)e^{3x} - e^{3x} - e^{3x} \log x$$

Q7 (a) Solve the differential equation $\frac{d^2 y}{dx^2} + 4y = x^2 + \cos 2x$

Answer

Q7 (b) Use method of variation of parameters to solve $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Answer

Q8 (a) Show that

$$(i) \int_0^{\pi/2} \sqrt{\sin \theta} d\theta \int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}} d\theta = \pi$$

$$(ii) \beta(m, n+1) + \beta(m+1, n) = \beta(m, n)$$

Answer

We know $B(m, m) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2x-1} \theta d\theta \dots \dots \dots (1)$

$\therefore \int_0^{\pi/2} \sqrt{\sin \theta} d\theta = \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$ by putting $m = \frac{3}{4}, x = \frac{1}{2}$ is....(1)

$\int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}} d\theta = \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$ by putting $m = \frac{1}{4}, x = \frac{1}{2}$ is....(1)

multiplying we are

$$\int_0^{\pi/2} \sqrt{\sin \theta} \int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}} d\theta = \frac{1}{4} \beta\left(\frac{3}{4}, \frac{1}{2}\right) \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\sqrt{\frac{3}{4}} \sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{4} + \frac{1}{2}}} \cdot \frac{\sqrt{\frac{1}{4}} \sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{4} + \frac{1}{2}}}$$

$$= \frac{1}{4} \frac{\sqrt{\frac{3}{4}} \sqrt{\frac{1}{2}}}{\sqrt{\frac{5}{4}}} \cdot \frac{\sqrt{\frac{1}{4}} \sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{4}}} = \frac{1}{4} \frac{\sqrt{\frac{1}{4}}}{\frac{1}{4} \sqrt{\frac{1}{4}}} \pi$$

$$U \sin g \sqrt{\frac{1}{2}} = \sqrt{\pi} = \pi$$

$$LHS = \beta(m, n+1) + \beta(m+1, n) = \frac{\sqrt{m}\sqrt{n+1}}{\sqrt{m+n+1}} + \frac{\sqrt{m+1}\sqrt{n}}{\sqrt{m+n+1}} = \frac{\sqrt{m^2\sqrt{n} + \sqrt{m}\sqrt{n}}}{\sqrt{m+n+1}}$$

$$= \frac{(m+n)\sqrt{m}\sqrt{n}}{(m+n)\sqrt{m+n}} = \frac{\sqrt{m}\sqrt{n}}{m+n} = \beta(m, n) = RHS$$

Q8 (b) Solve in series the equation $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

Answer

Here $x=0$ is an singular point because coefficient of $\frac{d^2y}{dx^2}$ become zero at $x=0$

\therefore Let $y = a_0 n^m + a_1 n^{m+1} + a_2 n^{m+2} + \dots + a_n n^{m+n} e \dots$

$\therefore \frac{dy}{dx} = ma_0 n^{m-1} + a_1(m+1)n^m + a_2(m+2)(m+1)n^m + \dots + a_n(m+n)(m+n-1)n^{m+n-2} \dots$

Substituting values in the given equation, we get

$$ax(1-x)[a_0 m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \dots + a_n(m+n)(m+n-1)x^{m+n-2}] - 12[a_0 mn^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \dots + a_n(m+n)n^{m+n-1} + \dots] + 14[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots + a_n x^{m+n} + \dots] = 0$$

Indicial equation is (equating to zero the coeff of low at powers of x

$$9a_0 m(m-1) - 12a_0 m = 0$$

i.e

$$m = 0, \frac{7}{3} \text{ because } a_0 \neq 0$$

\therefore Roots are distinct and do not differ by an integer.

equating coeff of different powers of x, we get

$$-9a_0 m(m-1) + 9a_1(m+1)m - 12a_1(m+1) + 4a_0 = 0$$

or

$$3a_1(m+1)(3m-4) = a_0 [9m^2 - 9m - 4]$$

$$= a_0(3m-4)(3m+1)$$

$$\therefore a_1 = \frac{3m+1}{3(m+1)} a_0$$

similarly

$$a_2 = \frac{3m+4}{3(m+2)} a_1 = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)} a_0$$

$$2fm = 0$$

$$a_1 = \frac{a_0}{3}, a_2 = \frac{4.1}{3^2 \cdot 2.1} a_0, a_3 = \frac{7 \times 4 \times 1}{3^3 \cdot 3 \cdot 2 \cdot 1} a_0$$

$$\therefore y_1 = a_0 \left(1 + \frac{1}{3}x + \frac{1.4}{3^2 \cdot 2} x^2 + \frac{1.4}{3^3 \cdot 3} x^3 e \dots \right)$$

$$\therefore y_2 = a_0 x^{\frac{7}{3}} \left(1 + \frac{8}{10}x + \frac{8.11}{10.13} x^2 + \frac{8.11.14}{10.13.16} x^3 e \dots \right)$$

Hence compute solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= A_0 \left(1 + \frac{1}{3}x + \frac{1.4}{3^2 \cdot 2} x^2 + \frac{1.4.7}{3^3 \cdot 3} x^3 e \right)$$

$$+ A_1 x^{\frac{7}{3}} \left(1 + \frac{8}{10}x + \frac{8.11}{10.13} x^2 + \frac{8.11.14}{10.13.16} x^3 e \dots \right)$$

Q9 (a) Show that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$

Answer

We know that

$$J_n(x) = \frac{x}{2x} [J_{n-1}(x) + J_{n+1}(x)]$$

$$\therefore J_{n+1}(x) = \frac{2x}{x} J_n(x) - J_{n-1}(x)$$

putting $n = 1, 2, 3$, we get

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x} \left[\frac{4}{x} J_2(x) - J_1(x) \right] - J_2(x) =$$

$$= \left(\frac{24}{x^2} - 1 \right) J_2(x) - \frac{6}{x} J_1(x)$$

$$= \left(\frac{24}{x^2} - 1 \right) \left(\frac{2}{x} J_1(x) - J_0(x) \right) - \frac{6}{x} J_1(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x)$$

Q9 (b) Show that $\int_{-1}^1 (1-x^2) P'_m(x) P'_n(x) dx = 0$

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Text Books

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi.

2. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd.