Q.2 a. Evaluate $\underset{x\to 1}{lt}\dfrac{x^x - x}{x-1-\log x}$ **(8)**

Answer:

Q.2 (a) $\lim\limits_{x\to 1} \dfrac{x^x - x}{x-1-\log x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x\to 1}{lt}\dfrac{\frac{d(x)^x}{dx} - 1}{1-\frac{1}{x}}$

Let $y = x^x$

$\log y = x\log x$

$\frac{1}{y}\frac{dy}{dx} = 1\cdot\log x + x\cdot\frac{1}{x}$

$\frac{dy}{dx} = x^x(1+\log x)$

$= \underset{x\to 1}{lt}\dfrac{x^x(1+\log x) - 1}{1-\frac{1}{x}}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \underset{x\to \phi}{lt}\dfrac{\frac{d}{dx}x^x, (1+\log x) + x^x\cdot\frac{1}{x} - 0}{\frac{1}{x^2}}$

$= \underset{x\to 1}{lt}\dfrac{x^x(1+\log x)^2 + x^{x-1}}{x^{-2}}$

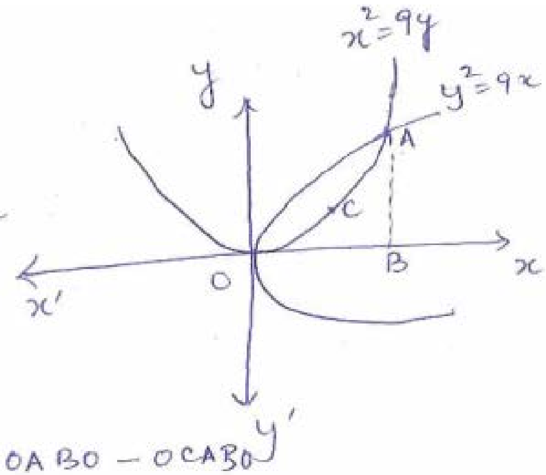
$= \dfrac{1.(1+0)^2 + 1}{1} = 2$

b. Find the area bounded by $y^2 = 9x$ **and** $x^2 = 9y$ **(8)**

Answer:

(b) $y^2 = 9x$ ——— ①

$x^2 = 9y$ ——— ②



Solving ① & ②, the points of intersection of the curves are O(0,0) and A(9,9)

Reqd. Area = OACO = Area OABO − OCABO

$$= \int_0^9 \sqrt{9x}\,dx - \int_0^9 \frac{x^2}{9}dx$$

$$= 3\int_0^9 \sqrt{x} - \frac{1}{9}\int_0^9 x^2\,dx$$

PTO →

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_0^9 - \frac{1}{9}\left[\frac{x^3}{3}\right]_0^9$$

$$= 2(27-0) - \frac{1}{3\times 9}[9\times 9\times 9 - 0]$$

$$= 54 - 27 = 27 \quad \text{Ans.}$$

Q.3 a. Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$ **(8)**

Answer:

Q. 3 (a) LHS

$= (1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$= \left(2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2\right)^n + \left(2\cos^2\theta/2 - 2i\sin\theta/2\cos\theta/2\right)^n$

$= 2^n\cos^n\theta/2\left(\cos\theta/2 + i\sin\theta/2\right)^n + 2^n\cos^n\theta/2\left(\cos\theta/2 - i\sin\theta/2\right)^n$

$= 2^n\cos^n\theta/2\left[\left(\cos\theta/2 + i\sin\theta/2\right)^n + \left(\cos\theta/2 - i\sin\theta/2\right)^n\right]$

$= 2^n\cos^n\theta/2\left[\cos n\theta/2 + i\sin n\theta/2 + \cos n\theta/2 - i\sin n\theta/2\right]$

$= 2^n\cos^n\theta/2 \cdot 2\cos n\theta/2$

$= 2^{n+1}\cos^n\theta/2\,\cos n\theta/2 = \text{RHS}$

b. **Forces of magnitudes 5 and 3 units acting in the directions $6i+2j+3k$ and $3i-2j+6k$ respectively act on a particle which is displaced from the point $(2,2,-1)$ to $(4,3,1)$. Find the work done by the forces. (8)**

Answer:

(b) $\vec{F}_1 = 5\left(\dfrac{6i+2j+3k}{\sqrt{36+4+9}}\right) = \dfrac{5}{7}(6i+2j+3k)$

$\vec{F}_2 = 3\,\dfrac{(3i-2j+6k)}{\sqrt{9+4+36}} = \dfrac{3}{7}(3i-2j+6k)$

Total Force $\vec{F} = \vec{F}_1 + \vec{F}_2$

$= \dfrac{1}{7}(39i+4j+33k)$

~~Dis~~

$\vec{d}$ = Displacement $= (4i+3j+k) - (2i+2j-k)$

$= 2i + j + 2k$

Work done $= \vec{F} \cdot \vec{d}$

$= \frac{1}{7}(39i + 4j + 33k) \cdot (2i + j + 2k)$

$= \frac{1}{7}(78 + 4 + 66)$

$= \frac{148}{7}$

Q.4 a. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. (8)

Answer:

Q. 4 (a) Let $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x - x^2 \, dx$

$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right]$

$= -\frac{1}{\pi} \cdot \frac{2\pi^3}{3} = -\frac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx \, dx.$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \int(1-2x)\frac{\sin nx}{n}dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - \left\{(1-2x)\left(-\frac{\cos nx}{n^2}\right) \mp (-2)\int -\frac{\cos nx}{n^2}dx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} + (1-2x)\frac{\cos nx}{n^2} + 2\,\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1-2\pi)(-1)^n}{n^2} - \frac{(1+2\pi)(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{\pi}\left(-\frac{4\pi}{n^2}\right) = \frac{-4(-1)^n}{n^2} = \frac{(-1)^{n+1}4}{n^2}$$

$\therefore\ a_1 = \frac{4}{1^2}$, $a_2 = -\frac{4}{2^2}$, $a_3 = \frac{4}{3^2}$, $a_4 = -\frac{4}{4^2}$ and so on.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{-(\pi-\pi^2)\frac{(-1)^n}{n} - \frac{2(-1)^n}{n^3} + (-\pi-\pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n}\right] = (-1)^{n+1}\cdot\frac{2}{n}$$

$b_1 = \frac{2}{1}$, $b_2 = -\frac{2}{2}$, $b_3 = \frac{2}{3}$, $b_4 = -\frac{2}{4}$ and so on.

$$\therefore \quad x - x^2 = \frac{-\pi^2}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

b. The differential equation for a circuit in which self-inductance neutralizes each other is $L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$ **. Find the current i as a function of t given that I is the maximum current, and i = 0 when t = 0.** **(8)**

Answer:

Q.4 (b)

$$L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2i}{dt^2} + \frac{i}{LC} = 0$$

or $\left(D^2 + \frac{1}{LC}\right)i = 0$ $\qquad D = \frac{d}{dt}$

A.E is $m^2 + \frac{1}{LC} = 0$

$$m = \pm \frac{j}{\sqrt{LC}} \qquad (j = \sqrt{-1})$$

$\therefore$ Soln. is

$$i = C_1 \sin\frac{1}{\sqrt{LC}}t + C_2 \cos\frac{1}{\sqrt{LC}}t \qquad \text{— (1)}$$

When $t=0$, $i=0$

$\therefore$ $C_2 = 0$

(1) becomes

$$i = C_1 \sin\frac{1}{\sqrt{LC}}t$$

Current i is maximum when $\sin\frac{1}{\sqrt{LC}}t = 1$

$\therefore$ $C_1 = I$

$\therefore$ Reqd. solution is $i = I \sin\frac{1}{\sqrt{LC}}t$

Q.5 a. Find the Laplace transform of $t^2 \cos at$. (8)

Answer:

Q.5(a)

$$\mathcal{L}(t^2 \cos at)$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)\left[-2s(s^2+a^2) - 4s(a^2-s^2)\right]}{(s^2+a^2)^4}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3} \quad \text{Ans.}$$

b. Evaluate $L^{-1}\left[\frac{s-1}{s^2-6s+25}\right]$ **(8)**

Answer:

(b)

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2-6s+25}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-1}{(s-3)^2+4^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+4^2} + \frac{2}{(s-3)^2+4^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+4^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{4}{(s-3)^2+4^2}\right]$$

$$= e^{3t}\cos 4t + \frac{1}{2}e^{3t}\sin \ldots$$

Q.6 a. Solve using Laplace transform the equation $\left(D^3-3D^2+3D-1\right)y=t^2e^t$, given that $y(0)=1, y'(0)=0, y''(0)=-2$. $\left(D=\frac{d}{dx}\right)$ **(8)**

Answer:

Q. 6 (a) $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$, $y(0)=1$, $y'(0)=0$, $y''(0)=-2$

Taking L.T of both sides, we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using given conditions

$$(s^3 - 3s^2 + 3s - 1)\bar{y} = s^2 + 2 - 3s + 3 + \frac{2}{(s-1)^3}$$

$$(s-1)^3\,\bar{y} = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y = L^{-1}\frac{1}{(s-1)} - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$$

$$= e^t - te^t - \frac{1}{2}t^2e^t + \frac{1}{60}e^t . t^5$$

$$= e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)$$

b. Solve $\dfrac{d^2y}{dx^2} - \dfrac{3dy}{dx} + 2y = xe^x$ **(8)**

Answer:

(b) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^x$$

A.E is

$$m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

C.F is $C_1e^x + C_2e^{2x}$

$$P.I = \frac{1}{D^2 - 3D + 2} xe^x$$

$$= e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} x$$

$$= e^x \frac{1}{D^2 - D} x$$

$$= -e^x \frac{1}{D}(1-D)^{-1} x$$

$$= -e^x \frac{1}{D}(1 + D + \cdots) x$$

$$= -e^x \frac{1}{D}(x+1)$$

$$= -e^x \left(\frac{x^2}{2} + x\right)$$

Complete solution is

$$y = C_1e^x + C_2e^{2x} - e^x\left(\frac{x^2}{2} + x\right)$$

Q.7 a. Using Maclaurin's series expand tan x upto the term containing x^5. (8)

Answer:

Q. 7 (a)

$$f(x) = \tan x \Rightarrow f(0) = 0$$

$$f'(x) = \sec^2 x = 1 + \tan^2 x \Rightarrow f'(0) = 1$$

$$f''(x) = 2\tan x \sec^2 x$$

$$= 2\tan x(1+\tan^2 x) \Rightarrow f''(0) = 0$$

$$f'''(x) = 2\sec^2 x(1+\tan^2 x) + 2\tan x(2\tan x \sec^2 x)$$

$$= 2(1+\tan^2 x)^2 + 4\tan^2 x(1+\tan^2 x)$$

$$= 6\tan^4 x + 8\tan^2 x + 2$$

$$\therefore f'''(0) = 2$$

$$f^{iv}(x) = 24\tan^3 x \sec^2 x + 16\tan x \sec^2 x \Rightarrow f^{iv}(0) = 0$$

$$= 24\tan^5 x + 40\tan^3 x + 16\tan x$$

$$f^{v}(x) = 120\tan^4 x \sec^2 x + 120\tan^2 x \sec^2 x + 16\sec^2 x \Rightarrow f^{v}(0) = 16$$

$\therefore$ Maclaurin's series is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\tan x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}16 + \cdots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

b. Using Laplace transforms evaluate the integral $\int_0^\infty te^{-2t}\sin t \, dt$ (8)

Answer:

7 (b)

$$\int_0^\infty t e^{-2t} \sin t \, dt = \int_0^\infty e^{-st} (t \sin t) \, dt \quad \text{where } s=2$$

$$= \mathcal{L}(t \sin t) \quad (\text{by def})$$

$$= (-1) \frac{d}{ds} \left(\frac{1}{s^2+1} \right)$$

$$= (-1) \frac{(-2s)}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$$

Put $s = 2$ back

$$\int_0^\infty t e^{-2t} \sin t \, dt = \frac{4}{25}$$

Q.8 a. Two circuits of impedances 2 + j4 ohms and 3 + j4 ohms are connected in parallel and a.c. voltage of 100 volts is applied across the parallel combination. Calculate the magnitude of the current as well as power factor for each circuit and the magnitude of the total current for the parallel combination. (8)

Answer:

Q. 8 (a) $\quad z_1 = 2+j4, \quad z_2 = 3+j4$

$$i_1 = \frac{V}{z_1} = \frac{100}{2+j4} = \frac{50}{1+2j} = \frac{50(1-2j)}{1+4} = 10(1-2j)$$

$$= 10 - 20j$$

$$|i_1| = \sqrt{100+400} = 10\sqrt{5} \text{ amp.}$$

$$\text{Power factor} = \frac{R_1}{|z_1|} = \frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}} = 0.447$$

$$i_2 = \frac{V}{Z_2} = \frac{100}{3+4j} = \frac{100(3-4j)}{25} = 4(3-4j)$$

$$= 12-16j$$

$$|i_2| = \sqrt{144+256} = 20 \text{ amp.}$$

$$\text{Power factor} = \frac{R_2}{|Z_2|} = \frac{3}{5} = .6$$

$$\text{Total current } i = i_1 + i_2$$

$$= (10-20j) + (12-16j)$$

$$= 22-36j$$

$$|i| = \sqrt{484+1296} = \sqrt{1780} = 2\sqrt{445} = 42.19$$

b. Express $f(x)=x$ as a Fourier series in the interval $-\pi < x < \pi$. (8)

Answer:

(b) Since $f(-x) = -x = -f(x)$, $f(x)$ is an odd function and therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \int\left(-\frac{\cos nx}{n}\right)dx \right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n} \right]$$

$$= (-1)^{n+1}\frac{2}{n}$$

$$b_1 = \frac{2}{1},\ b_2 = -\frac{2}{2},\ b_3 = \frac{2}{3},\ b_4 = -\frac{2}{4} \cdots$$

$$\therefore\ x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots \right)$$

Q.9 a. If $(x^2y-2)+i(x+2xy-5)=0$, find the value of x and y. (8)

Answer:

Q. 9. (a) $(x^2y-2)+i(x+2xy-5)=0$

Equating real & imaginary parts

$\Rightarrow x^2y-2=0$ — (1) & $x+2xy-5=0$ — (2)

(1) gives

$y=\frac{2}{x^2}$ — (3)

Substituting in (2)

$x+2x.\frac{2}{x^2}-5=0$

or $x^2+4-5x=0$

$(x^2-5x+4)=0$

$(x-4)(x-1)=0$

$x=1$ or $x=4$.

Case 1. When $x=1$

$y=2$ (from (3))

Case 2 When $x=4$

$y=\cancel{\frac{1}{2}}\ \frac{1}{8}$ (from (3))

MODERATION-I

b. Evaluate $\int_0^{\pi/6} cos^4 3\theta\, sin^3 6\theta\, d\theta$ **(8)**

Answer:

Q. 9(b). $\displaystyle\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$

$$= \int_0^{\pi/6} \cos^4 3\theta \left(2\sin 3\theta \cos 3\theta\right)^3 d\theta$$

$$= 8\int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta \, d\theta$$

Put $3\theta = x$

$3\,d\theta = dx \Rightarrow d\theta = \frac{dx}{3}$

& for $\theta = 0$, $x = 0$, & for $\theta = \pi/6$, $x = \frac{\pi}{2}$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^3 x \cos^7 x \, dx$$

$$= \frac{8}{3} \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\,\Gamma\left(\frac{3+7+2}{2}\right)}$$

$$= \frac{8}{3} \cdot \frac{1.3.2.1}{2.5.4.3.2.1}$$

$$= \frac{1}{15}$$

OR $\displaystyle\frac{8}{3} \cdot \frac{2\times 6\cdot 4\cdot 2}{10.8.6.4.2} = \frac{1}{15}$

(using $\displaystyle\frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-2)\ldots}$

<u>TEXT BOOK</u>

I. Engineering mathematics –Dr. B.S.Grewal, 12th edition 2007, Khanna publishers, Delhi

II. Engineering Mathematics – H.K.Dass, S. Chand and Company Ltd, 13th Revised Edition 2007, New Delhi

III. A Text book of engineering Mathematics – N.P. Bali and Manish Goyal , 7th Edition 2007, Laxmi Publication(P) Ltd