Q.2 a. If $x\sqrt{1+y}+y\sqrt{1+x}=0$**, then prove that** $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$ **(8)**

Answer:

Q.2. (a)

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$x\sqrt{1+y}=-y\sqrt{1+x} \quad (1)$$

Squaring both sides, we get

$$x^2(1+y)=y^2(1+x)$$

$$x^2+x^2y=y^2+xy^2$$

$$x^2-y^2+x^2y-xy^2=0 \quad (1)$$

$$(x-y)(x+y)+xy(x-y)=0$$

$$(x-y)[x+y+xy]=0 \quad (2)$$

$$x-y\neq 0, \therefore x+y+xy=0 \quad (1)$$

$$y(1+x)=-x$$

$$y=\frac{-x}{1+x} \quad (1)$$

Differentiating both sides, w.r.t. 'x'

$$\frac{dy}{dx}=-\left[\frac{(1+x)\frac{d}{dx}x-x\cdot\frac{d}{dx}(1+x)}{(1+x^2)}\right]$$

$$=-\left[\frac{(1+x)\cdot 1-x(1)}{(1+x)^2}\right]$$

$$=-\left[\frac{1+\cancel{x}-\cancel{x}}{(1+x)^2}\right]=\frac{-1}{(1+x)^2} \quad (2) \text{ Ans.}$$

b. Find the point on the curve $y=7x-3x^2$**, where the inclination of the tangent with x-axis is of** 45°**. Also find the equation of the normal to the given curve at that point.** **(8)**

Answer:

b. Let $P(x_1, y_1)$ be a point on the curve $y = 7x - 3x^2$ where the tangent is inclined at an angle of 45^0 with the x-axis, Then slope of the tangent at $P = \tan 45^0$

i.e. $\left(\frac{dy}{dx}\right)_P = \tan 45^0$ (1) $\Rightarrow \left(\frac{dy}{dx}\right)_P = 1$ (1) —— (i)

Differentiating $y = 7x - 3x^2$, w.r.t. 'x', we get

$\frac{dy}{dx} = 7 - 6x$ (1)

$\therefore \left(\frac{dy}{dx}\right)_P = 7 - 6x_1$ —— (ii)

from (i) and (ii), we get

$7 - 6x_1 = 1 \Rightarrow x_1 = 1$ (1)

Since $P(x_1, y_1)$ lies on the curve $y = 7x - 3x^2$

$\therefore y_1 = 7x_1 - 3x_1^2$

$\Rightarrow y_1 = 7 - 3 = 4$ (1) [Putting $x_1 = 1$]

Hence, the required point on the curve is $(1, 4)$ Ans

Slope of normal $= -1$ (1) [say m]

$\therefore$ Equation of normal is

$y - y_1 = m(x - x_1)$ (1)

$\Rightarrow y - 4 = -1(x - 1)$

$\Rightarrow y - 4 = -x + 1$

$\Rightarrow x + y = 5$ (1)

Q.3 a. Evaluate $\int x\cos^3 x dx$ **(8)**

Answer:

Q. 3. a.

$$\int x\cos^3 x\,dx$$

$$= \int x\left(\frac{1}{4}\cos 3x + \frac{3}{4}\cos x\right)dx \quad (1)$$

$$= \frac{1}{4}\int \underset{I}{x}\,\underset{II}{\cos 3x}\,dx + \frac{3}{4}\int \underset{I}{x}\,\underset{II}{\cos x}\,dx \quad (1)$$

$$= \frac{1}{4}\left[x\int \cos 3x\,dx - \int\left(\frac{d}{dx}x\int \cos 3x\right)dx\right] +$$

$$+\frac{3}{4}\left[x\int \cos x\,dx - \int\left(\frac{d}{dx}x\int \cos x\,dx\right)dx\right] +$$

$$= \frac{1}{4}\left[\frac{x\sin 3x}{3} - \int \frac{\sin 3x}{3}dx\right] \; (1) + \frac{3}{4}\left[x\sin x - \int \sin x\,dx\right] \; (1) + C$$

$$= \frac{1}{4}\left[\frac{x\sin 3x}{3} + \frac{1}{3}\left(\frac{\cos 3x}{3}\right)\right] + \frac{3}{4}\left[x\sin x - (-\cos x)\right] + C$$

$$= \frac{1}{4}\left[\frac{x\sin 3x}{3} + \frac{\cos 3x}{9}\right] \; (2) + \frac{3}{4}\left[x\sin x + \cos x\right] \; (2) + C \quad \text{Ans.}$$

b. Evaluate $\int_0^{\pi/2} x^2\cos^2 x\,dx$ **(8)**

Answer:

b.

$$\int_0^{\pi/2} x^2\cos^2 x\,dx$$

$$= \int_0^{\pi/2} x^2\cdot\frac{(1+\cos 2x)}{2}dx \quad (1)$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2\,dx + \frac{1}{2}\int_0^{\pi/2} x^2\cos 2x\,dx \quad (1)$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^{\pi/2} (1) + \frac{1}{2}\left[\left[x^2\frac{\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} 2x\frac{\sin 2x}{2}dx\right] \; (1) \; (2)$$

$$= \frac{1}{6}\left(\frac{\pi^3}{8} - 0\right) (1) + \frac{1}{2}\left[0 - \int_0^{\pi/2} x\sin 2x\,dx\right] \; (1)$$

$$= \frac{\pi^3}{48} - \frac{1}{2}\int_0^{\pi/2} x\cdot\sin 2x\,dx$$

$$= \frac{\pi^3}{48} - \frac{1}{2}\left[\left[-x\frac{\cos 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\left(-\frac{\cos 2x}{2}\right)\right]$$

$$= \frac{\pi^3}{48} - \frac{1}{2}\left[\left(-\frac{\pi}{4}\cos\pi - 0\right) + \frac{1}{4}\left[\sin 2x\right]_0^{\pi/2}\right]$$

$$= \frac{\pi^3}{48} - \frac{1}{2}\left[\frac{\pi}{4} + 0\right] = \frac{\pi^3}{48} - \frac{\pi}{8} \quad (2) \quad \underline{\text{Ans}}$$

Q.4 a. Solve $\cos(x+y)dy = dx$ **(8)**

Answer:

Q.4(a). Soln:

$\cos(x+y)\,dy = dx$

or $\frac{dy}{dx} = \sec(x+y)$

On putting $x+y = z$ (1)

So that, $1 + \frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = \frac{dz}{dx} - 1$

or $\frac{dz}{dx} - 1 = \sec z$ (1)

or $\frac{dz}{dx} = 1 + \sec z$

or $\frac{dz}{dx} = 1 + \sec z$

or $\frac{dz}{1+\sec z} = dx$ (1)

$\int \frac{\cos z}{\cos z + 1}\,dz = \int dx + c$

or $\int \left[1 - \frac{1}{\cos z + 1}\right] dz = x + c$ (1)

or $\int \left[1 - \frac{1}{2\cos^2\frac{z}{2} - 1 + 1}\right] dz = x + c$

or $\int \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right) dz = x + c$ (1)

or $z - \tan\frac{z}{2} = x + c$ (2)

or $x + y - \tan\frac{x+y}{2} = x + c$

or $y - \tan\frac{x+y}{2} = c$ (1) Ans.

b. Solve $\frac{dy}{dx} + y \cdot \sec x = \tan x$ **(8)**

Answer:

Q.4.b. Soln. The given differential eqn. is,

$$\frac{dy}{dx} + y\sec x = \tan x \quad \text{---} (1)$$

This is a linear differential eqn. of the form,

$\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$

$$\therefore \text{I.F.} = e^{\int P dx} (1) = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)} (1)$$

$$= (\sec x + \tan x) (1)$$

Multiplying both sides of (1) by I.F. $= (\sec x + \tan x)$,

we get, $(\sec x + \tan x)\frac{dy}{dx} + y\sec x(\sec x + \tan x)$
$= \tan(\sec x + \tan x)$

Integrating both sides, w.r.t. 'x' we get

$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\, dx (1) + c$

$\left[\text{using } y(\text{I.F.}) = \int Q.(\text{I.F.}) (1)\, dx + c\right]$

$\Rightarrow y(\sec x + \tan x) = \int(\tan x \cdot \sec x + \tan^2 x)\, dx + c$

$= \int(\tan x \cdot \sec x + \sec^2 x - 1) (1)\, dx + c$

$= \sec x + \tan x - x + c$. Ans.

(1) (1)

5

Q.5 a. Find the term independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$ **(8)**

Answer:

Q.5.a. Soln:

$(2x^2 - \frac{1}{x})^{12}$

Here $n = 12$, $x \to 2x^2$, $a \to -\frac{1}{x}$

$T_{r+1} = {}^nC_r\, x^{n-r} a^r$ (General Term)

$= {}^{12}C_r\, (2x^2)^{12-r} \cdot \left(\frac{-1}{x}\right)^r$

$= {}^{12}C_r\, (2)^{12-r}\, x^{24-2r} \cdot \frac{(-1)^r}{x^r}$

$= {}^{12}C_r\, (2)^{12-r}\, x^{24-3r}\, (-1)^r$

To find the term independent of 'x' put $24 - 3r = 0$

$r = 8$

$T_{8+1} = {}^{12}C_8\, (2)^{12-8} \cdot x^{24-24} \cdot (-1)^8$

$= \frac{12!}{8!\,4!} (2)^4\, x^0\, (1)$

$= \frac{12 \times 11 \times 10 \times 9 \times 8!}{8! \times 4 \times 3 \times 2 \times 1} \times 16 = 7920$ Ans.

b. The sum of first three terms of a G.P. is 16 and the sum of the next three term is 128. Find the sum of 1st n terms (S_n) of G.P. (8)

Answer:

Q.5.b. Soln:

Let a be the first term and r the common ratio of the G.P., then,

$a + ar + ar^2 = 16$ — (i)

and $ar^3 + ar^4 + ar^5 = 128$ — (ii)

$\Rightarrow a(1 + r + r^2) = 16$ and $ar^3(1 + r + r^2) = 128$

$$\Rightarrow \frac{ar^3(1+r+r^2)}{a(1+r+r^2)} = \frac{128}{16} \quad 8$$

$$\Rightarrow r^3 = 8 \quad (1) \Rightarrow r = 2 \quad (1)$$

Putting $r=2$ in (1), we get $a = \frac{16}{7}$ (1)

$$\therefore S_n = a\left(\frac{r^n-1}{r-1}\right) \quad (1)$$

$$\Rightarrow S_n = \frac{16}{7}\left(\frac{2^n-1}{2-1}\right) = \frac{16}{7}(2^n-1) \quad (2) \text{ Ans.}$$

Q.6 a. Prove that, $\cos 2A \cdot \cos 2B + \sin^2(A-B) - \sin^2(A+B) = \cos(2A+2B)$ **(8)**

Answer:

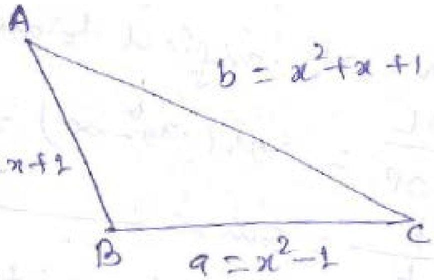
Q.6.a. Soln.

$$\text{LHS} = \cos 2A \cdot \cos 2B + \sin^2(A-B) - \sin^2(A+B)$$
$$= \cos 2A \cdot \cos 2B + \sin(A-B+A+B) \cdot \sin(A-B-A-B) \quad (2)$$
$$\left[\because \sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B\right]$$
$$= \cos 2A \cdot \cos 2B + \sin 2A \cdot \sin(-2B) \quad (2) \quad [\because \sin(-\theta) = -\sin\theta$$
$$= \cos 2A \cdot \cos 2B - \sin 2A \cdot \sin 2B$$
$$= \cos(2A + \cos 2B) \quad (4)$$
$$= \text{RHS. Hence proved.}$$

b. The sides of a triangle are $x^2+x+1, 2x+1$ **and** x^2-1. **Find the greatest angle.** **(8)**

Answer:

Q.6. b. Soln.



The given sides of a triangle are x^2+x+1, $2x+1$ and x^2-1.

Since, $AC = x^2+x+1$ is the largest side, therefore, the angle opposite AC is largest i.e. angle B is the largest angle (1)

$$\cos B = \frac{a^2+c^2-b^2}{2ac} \ (1) = \frac{(x^2-1)^2+(2x+1)^2-(x^2+x+1)^2}{2(x^2-1)\ (2x+1)}$$

$$= \frac{x^4+1-2x^2+4x^2+1+4x-(x^4+x^2+1+2x^3+2x+2x^2)}{2(2x^3+x^2-2x-1)}$$

$$= \frac{\cancel{x^4}+2+2x^2+4x-\cancel{x^4}-x^2-1-2x^3-2x-2x^2}{2(2x^3+x^2-2x-1)}$$

$$= \frac{-2x^3-x^2+2x+1}{2(2x^3+x^2-2x-1)} \ (2) = \frac{-(2x^3+x^2-2x-1)}{2(2x^3+x^2-2x-1)} = -\frac{1}{2}$$

$$= -\frac{1}{2}$$

$$\cos B = \frac{-1}{2}$$

$$\cos B = \cos(180^\circ - 60^\circ)$$

$$B = 120^\circ \ (1)$$

Ans

Q.7 a. Find the equation of the circle which passes through the points $(3,-2),(-2,0)$ and having its centre on the line 2x – y = 3. (8)

Answer:

Q.7.a. Soln:

Let the eqn of the circle be,

$x^2+y^2+2gx+2fy+c=0$ 1 —— (i)

As (i) passes through $(3,-2)$

$\therefore 9+4+6g-4f+c=0$

or $6g-4f+c=-13$ 1 —— (ii)

Also (i) passes through $(-2,0)$,

$\therefore 4+0-4g-0+c=0$

or $4g-c=4$ 1 —— (iii)

The Centre $(-g,-f)$ of (i) lies $2x-y=3$

$\therefore -2g+f=3$ 1 —— (iv)

Adding (ii) and (iii), we get

$10g-4f=-9$ —— (v)

Solving (iv) and (v), we get

$g=3/2$ 1, $f=6$ 1

Putting g in (iii), we get $c=2$ 1

Substituting these values of g, f and c in (i), we get

$x^2+y^2+3x+12y+2=0$ 1 Which is the reqd. eqn of the circle.

Ans

b. Find the vertex, focus, directrix, axis and latus-rectum of the parabola of $y^2=4x+4y$ **(8)**

Answer:

Q.9.b. Soln:

The given eqn. is $y^2 = 4x + 4y$

or $y^2 - 4x = 4y$ or $y^2 - 4y + 4 = 4x + 4$

or $(y-2)^2 = 4(x+1)$ 2 —— (i)

Shifting the origin to the point $(-1, 2)$ without rotating the axes and denoting the new co-ordinates with respect to these axes by X and Y, we have

$$\left.\begin{array}{l} x = X + (-1) \\ y = Y + 2 \end{array}\right] \text{—— (ii)}$$

Using these relation eqn. (i), reduce to,

$\boxed{Y^2 = 4X}$ 1

This is the form of $Y^2 = 4a \cdot X$ on comparing, we get $4a = 4$ or $a = 1$. The co-ordinates of the vertex w.r.t. new axes are $(X = 0, Y = 0)$

∴ Co-ordinates of the vertex w.r.t. old axes are $(-1, 2)$ 1 [Putting $X = 0$ and $Y = 0$ in (ii)]

The coordinates of the focus w.r.t. new axes are $(X = 0, Y = 0)$

∴ Co-ordinate of the focus w.r.t. old axes are $(0, 2)$ [∵ Putting $X = 1, Y = 0$ in (ii)] 1

Eqn. of the directrix of the parabola w.r.t. new axes is $X = -1$.

∴ The eqn. of the directrix of parabola w.r.t. old axes is $x = -2$ 1 [Putting $X = -1$ in eqn (ii)]

7(b)

~~Q.9.b.~~ Soln.

Eqn. of the axes of the parabola w.r.t. new axes is $Y \cancel{y} = 0$ cap.

∴ Eqn. of axes w.r.t. old axes is $\cancel{x}\, y = 2$ 1

[Putting $X = 0$ in (ii)]

The length of Latus rectum $= 4$ 1 Ans

Q.8 a. If p be the length of perpendicular from the origin to the line whose intercepts on the axes are a and b respectively, then show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$ **(8)**

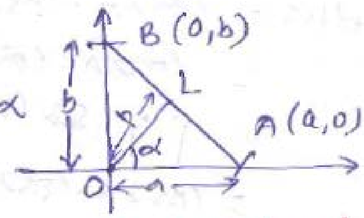
Answer:

Q.8.a. Soln:

Let AB be the line which cuts co-ordinate axes at A & B such that OA = a, and OB = b

Let OL ⊥ AB and OL = p

then ∠AOL = α then ∠LOB = $90^\circ-\alpha$



In rt. angled triangle ALO,

$$\frac{OL}{OA}=\cos\alpha$$

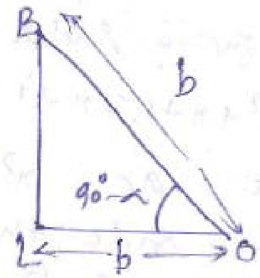
$$\frac{p}{a}=\cos\alpha \quad (2) \longrightarrow (i)$$

(2) [For figure]

In rt. angled triangle BOL, we have

$$\frac{OL}{OB}=\cos(90^\circ-\alpha)=\sin\alpha$$

$$\frac{p}{b}=\sin\alpha \quad (2) \longrightarrow (ii)$$



Squaring and adding (i) and (ii)

$$\frac{p^2}{a^2}+\frac{p^2}{b^2}=\cos^2\alpha+\sin^2\alpha \quad (1)$$

$$\frac{p^2}{a^2}+\frac{p^2}{b^2}=1$$

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$$

OR $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$ (1) Hence proved

b. Find the equation of the straight lines through the point $(2,-1)$ and making an angle of 45° with the line $6x+5y-1=0$ **(8)**

Answer:

8(b)

Q.8.b Soln:

The eqn. of st. line through (2,−1) and having slope m is $y+1 = m(x-2)$ —— (i) (2)

(i) makes an angle 45° with line $6x+5y-1=0$ —— (ii)

Slope of (ii) is $-\frac{6}{5}$ (1)

$\tan 45^\circ = \pm \dfrac{m+\frac{6}{5}}{1-\frac{6m}{5}}$ (1)

or $1 = \pm \dfrac{5m+6}{5-6m}$

Taking the +ve sign.

$5-6m = +(5m+6) \Longrightarrow m = -\frac{1}{11}$ (1)

Taking −ve sign.

$5-6m = -(5m+6) \Longrightarrow m = 11$ (1)

Putting the values of m in (i) we get

$y+1 = -\frac{1}{11}(x-2)$ and $y+1 = 11(x-2)$

or $x+11y+9=0$ (1) and $11x-y-23=0$ (1) are the required eqn.s of the st. lines. Ans

Q.9 a. **If** $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$**, find** A^{-1} **and show that** $A^{-1} = A^2$ **(8)**

Answer:

9(a)

Q.4.a. Soln:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For A^{-1}, $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$$|A| = 1(0-0) + 1(0-0) + 1(0+1) = 1 \neq 0 \quad (1)$$

$\therefore A^{-1}$ exists,

$A_{11} = +(0-0) = 0, \quad A_{21} = -(0-0) = 0, \quad A_{31} = +(0+1) = 1$

$A_{12} = -(0+0) = 0, \quad A_{22} = +(0-1) = -1, \quad A_{32} = -(0-2) = 2 \quad (3)$

$A_{13} = +(0+1) = 1, \quad A_{23} = -(0+1) = -1, \quad A_{33} = +(-1+2) = 1$

1 marks for 3 correct A_{ij}

$$\text{adj}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad (1)$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{1}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad (1) \longleftarrow (i)$$

$$A^2 = A \times A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)-1(2)+1(1) & 1(-1)-1(-1)+1(0) & 2(1)-1(0)+1(0) \\ 2(1)-1(2)+0(1) & 2(-1)-1(-1)+0(0) & 2(1)-1(0)+0(0) \\ 1(1)+0(2)+0(1) & 1(-1)+0(-1)+0(0) & 1(1)+0(0)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1-0+0 \\ 2-2+0 & -2+1+0 & 2-0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad (1) \longrightarrow (ii)$$

From (i) and (ii) $\longrightarrow A^2 = A^{-1}$ (1)

Hence proved.

b. If $\begin{bmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{bmatrix} = 0$. **Prove that abc = 1.** **(8)**

Answer:

9 (b)
$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0 \quad \text{or}$$

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = 0 \quad \text{---} \quad 2$$

$$\Rightarrow abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0 \quad \text{---} \quad 1$$

$$\Rightarrow (abc-1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \quad \text{---} \quad 4$$

$$\text{---} \quad 1$$

$$\Rightarrow abc = 1$$

TEXT BOOK

I. Applied Mathematics for Polytechnics, H. K. Dass, 8th Edition, CBS Publishers & Distributors

II. A Text book of Comprehensive Mathematics Class XI, Parmanand Gupta, Laxmi Publications (P) Ltd, New Delhi

III. Engineering Mathematics, H. K. Dass, S, Chand and Company Ltd, 13th Edition, New Delhi