Q.2 a. Expand $\log_e x$ in powers of (x-1) and hence evaluate $\log_e 1.1$ correct to 4 places of decimal. (8)

Answer:

Q.2 (a) Let $f(x) = \log_e x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$

$f^{iv}(x) = -6/x^4$ $f^{iv}(1) = -6$

and so on.

By Taylor's series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

Or Replacing x by a & h by $x-a$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

Taking $a = 1$

$$f(x) = f(1) + \frac{(x-1)}{1!}f'(1) + \cancel{(x-1)^2}\frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots$$

Substituting the values of f, f', f'' & f''' at $x = 1$

$$\log_e x = f(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \ldots$$

$$\log_e x = f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Putting $x = 1.1$, so that $x-1 = .1$, we have

$$\log 1.1 = .1 - \frac{1}{2}(.1)^2 + \frac{1}{3}(.1)^3 - \frac{1}{4}(.1)^4 + \ldots$$

$$= .1 - .005 + .0003 - .00002 + \ldots$$

$$= \cancel{.095275} \quad .0953$$

b. Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (8)

Answer:

2. (b) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$ ($\infty - \infty$ form)

$= \lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x} \right)$ ($\frac{0}{0}$ form)

$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x}$ ($\frac{0}{0}$ form)

$= \lim_{x \to 0} \frac{\sin x}{x(-\sin x) + \cos x + \cos x}$.

$= \frac{0}{0+1+1} = 0.$

Q.3 a. Evaluate by using reduction formula **(8)**

$\int_0^{\pi/6} cos^4 3\phi \, sin^2 6\phi d\phi$

Answer:

Q. 3 (a) $$\int_0^{\pi/6} \cos^4 3\phi \sin^2 6\phi \, d\phi$$

$$= \int_0^{\pi/6} \cos^4 3\phi \left(2\sin 3\phi \cos 3\phi\right)^2 d\phi$$

$$= 4\int_0^{\pi/6} \cos^6 3\phi \sin^2 3\phi \, d\phi$$

Put $3\phi = \theta$

$3\,d\phi = d\theta$ or $d\phi = \frac{d\theta}{3}$

when $\phi = 0$, $\theta = 0$, when $\phi = \frac{\pi}{6}$, $\theta = \pi/2$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^6\theta \sin^2\theta \, d\theta$$

(Here m & n are both even)

$$= \frac{4}{3} \frac{(6-1)(6-3)(6-5)\,.\,(2-1)}{8.(8-2)(8-4)(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{\cancel{4}}{\cancel{3}} \frac{5.\cancel{3}\,1\,.\,1}{8.6.\cancel{4}.2} \cdot \frac{\pi}{2}$$

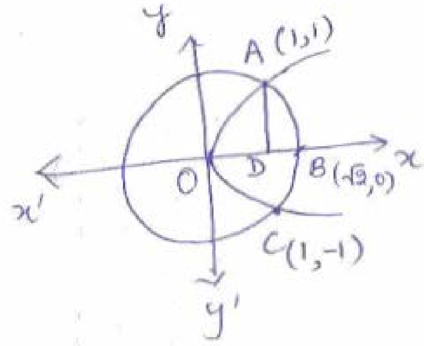
$$= \frac{5\pi}{192}$$

b. Find the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$. **(8)**

Answer:

Q.3 (b) $y^2 = x$ —— (1)

$x^2+y^2 = 2$ —— (2)



Solving (1) & (2)

$x^2 + x = 2$

or $x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1, -2$

$\therefore y = \pm 1$, or $\pm i\sqrt{2}$

$\therefore$ Points of intersection of (1) & (2) are $A(1,1)$ & $C(1,-1)$

B has coordinates $(\sqrt{2}, 0)$

Reqd. Area $=$ Ar OABCO $= 2$ ar. OABDO

$= 2\left[\text{ar OADO} + \text{ar ABDA}\right]$

$= 2\left[\int_0^1 \sqrt{x}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx\right]$

$$= 2\left[\left(\frac{2}{3}x^{3/2}\right)_0^1 + \left(\frac{x}{2}\sqrt{2-x^2} + \frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right)_1^{\sqrt{2}}\right]$$

$$= 2\left[\frac{2}{3} + \sin^{-1}1 - \frac{1}{2} - \sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\left[\frac{2}{3} + \frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right]$$

$$= \frac{4}{3} + \pi - \frac{\pi}{2} - 1$$

$$= \cancel{\frac{4}{3}} + \frac{1}{3} + \frac{\pi}{2}$$

Q.4 a. If *n* is positive integer, prove that

$$\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = 2^{n+1}\cos\frac{n\pi}{6},\quad \left(i=\sqrt{-1}\right)$$ **(8)**

Answer:

Q. 4 (a) Let $\sqrt{3}+i = r(\cos\alpha + i\sin\alpha)$

$$r = \sqrt{3+1} = 2, \quad \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n + 2^n\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^n$$

$$= 2^n\left[\left(\cos n\frac{\pi}{6} + i\sin n\frac{\pi}{6}\right) + \left(\cos n\frac{\pi}{6} - i\sin n\frac{\pi}{6}\right)\right]$$

$$= 2^n \cdot 2\cos n\frac{\pi}{6}$$

$$= 2^{n+1}\cos n\frac{\pi}{6}$$

b. Two impedances $Z_1 = 8 + j6$ ohms & $Z_2 = 6 - j8$ ohms are connected in parallel across 200 volts, calculate the magnitude of current in each branch and the total current in the circuit. (8)

Answer:

Q.4 (b) $Z_1 = 8+6j$ & $Z_2 = 6-8j$

$$i_1 = \frac{V}{Z_1} = \frac{200}{8+6j}$$

$$= \frac{200(8-6j)}{(8+6j)(8-6j)}$$

$$= 4(4-3j)$$

magnitude of $i_1 = 4\sqrt{16+9} = 20$ amperes.

$$i_2 = \frac{V}{Z_2} = \frac{200}{6-8j}$$

$$= \frac{200(6+8j)}{(6-8j)(6+8j)}$$

$$= 4(3+4j)$$

Magnitude of $i_2 = 4\sqrt{9+16} = 20$ amperes.

Total current $i_1 + i_2 = 4(4-3j) + 4(3+4j)$

$$= 28 + 4j$$

Magnitude of total current $= \sqrt{28^2+4^2} = \sqrt{800}$

$$= 20\sqrt{2} \text{ amperes}$$

Q.5 a. Forces of magnitude 5 and 3 units acting in the directions $6i + 2j + 3k$ and $3i - 2j + 6k$ respectively act on a particle which is displaced from the point (2,2,-1) to (4,3,1). Find the work done by the forces. (8)

Answer:

Q.5. (a) $\vec{F}_1 = \frac{5(6i+2j+3k)}{\sqrt{36+4+9}} = \frac{5}{7}(6i+2j+3k)$

$\vec{F}_2 = \frac{3(3i-2j+6k)}{\sqrt{9+4+36}} = \frac{3}{7}(3i-2j+6k)$

Total force $\vec{F} = \vec{F}_1 + \vec{F}_2$

$= \frac{5}{7}(6i+2j+3k) + \frac{3}{7}(3i-2j+6k)$

$= \frac{1}{7}(39i+4j+33k)$

Displacement $\vec{d} = (4i+3j+k) - (2i+2j-k)$

$= 2i+j+2k$

Work done $= \vec{F}.\vec{d}$

$= \frac{1}{7}(39i+4j+33k).(2i+j+2k)$

$= \frac{1}{7}(78+4+66)$

$= \frac{148}{7}$ units

b. Find the volume of the parallelepiped, if $\vec{a} = -3i + 7j - 5k$, $\vec{b} = -3i + 7j - 3k$ and $\vec{c} = 7i - 5j - 3k$ are the three coterminous edges of the parallelepiped. (8)

Answer:

Q.5. (b)

$$\text{Volume} = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} -3 & 7 & -5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= -3(-21-15) - 7(9+21) - 5(15-49)$$

$$= 108 - 210 + 170$$

$$= 68$$

Q.6 **a. Solve** $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin 2x$ **(8)**

Answer:

Q. 6. (a) $\dfrac{d^2y}{dx^2} + \dfrac{3dy}{dx} + 2y = \sin 2x$

A.E is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1, -2.$$

C.F. is $C_1e^{-x} + C_2e^{-2x}$.

$$P.I = \frac{1}{D^2+3D+2}\sin 2x$$

$$= \frac{1}{-4+3D+2}\sin 2x \qquad (D^2=-4)$$

$$= \frac{1}{3D-2}\sin 2x$$

$$= \frac{(3D+2)}{9D^2-4}\sin 2x$$

$$= -\frac{1}{40}(3D+2)\sin 2x$$

$$= -\frac{1}{40}\left[6\cos 2x + 2\sin 2x\right]$$

$$= -\frac{1}{20}\left[3\cos 2x + \sin 2x\right]$$

Complete solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

b. A Condenser of capacity *C* is discharged through the inductance *L* and a resistance *R* in series and the charge *q* at any time *t* satisfies the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

Given that L = 0.25 henry, R = 250 ohms, $C = 2\times10^{-6}$ farad and that when t = 0, the charge q=0.002 coulombs, and the current $\frac{dq}{dt} = 0$, obtain the value of *q* in terms of *t*. (8)

Answer:

 266/4

Q. 6.(b) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0$

Substituting the given values

$$\frac{d^2q}{dt^2} + \frac{250}{.25}\frac{dq}{dt} + \frac{1}{2\times10^{-6}\times .25}q = 0$$

or $$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + \cancel{50\times}\ 2\times10^6\ q = 0$$

A.E is $$m^2 + 1000m + 2\times10^6 = 0$$

$$m = \frac{-1000 \pm \sqrt{10^6 - 8\times10^6}}{2}$$

$$= -500 \pm 500i\sqrt{7} = -500 \pm 1323i$$

∴ Solution is

$$q = e^{-500t}\left(A\cos 1323t + B\sin 1323t\right)$$

When $t=0$, $q = 0.002$

∴ $A = .002$

$$\frac{dq}{dt} = -500e^{-500t}(A\cos 1323t + B\sin 1323t) + e^{-500t}(-1323A\sin 1323t + 1323B\cos 1323t)$$

When $t=0$, $\frac{dq}{dt} = 0$

$$0 = -500A + 1323B$$

$$\Rightarrow B = \frac{500\times .002}{1323} = .0008$$

$$\therefore q = e^{-500t}\left(.002\cos 1323t + .0008\sin 1323t\right)$$

Q.7 Test for convergence the series given below:

a. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ **(8)**

Answer:

Q. 7 (a) Here $u_n = \frac{n!}{n^n}$

Applying Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{(n+1) \cdot n!}{(n+1) \cdot (n+1)^n} \cdot \frac{n^n}{n!}$$

$$= \frac{n^n}{(n+1)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Hence the given series is convergent.

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ **(8)**

Answer:

(b)

$$u_n = \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

$$= \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})}$$

$$= \sqrt{n+1} - \sqrt{n}.$$

$$= \sqrt{n}\left[\left(1+\frac{1}{n}\right)^{1/2} - 1\right]$$

$$= \sqrt{n}\left[\left(1+\frac{1}{2n} - \frac{1}{8n^2} + \cdots\right) - 1\right] \quad \text{(using binomial expansion)}$$

$$= \sqrt{n}\left(\frac{1}{2n} - \frac{1}{8n^2} + \cdots\right)$$

$$u_n = \frac{1}{\sqrt{n}}\left(\frac{1}{2} - \frac{1}{8n} + \cdots\right)$$

Taking $v_n = \frac{1}{\sqrt{n}}$, we have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{u_n}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{8n} + \cdots\right)$$

$= \frac{1}{2}$. which is finite and non-zero.

$\therefore$ $\sum u_n$ & $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is known to be divergent

Hence $\sum u_n$ is also divergent.

Q.8 Find the Laplace transform of $f(t)$ where

a. $f(t) = 2\sin 2t \cos 4t$ **(8)**

Answer:

Q. 8 (a) $f(t) = 2\sin 2t \cos 4t$

$$= \sin(2t+4t) + \sin(2t-4t)$$

$$= \sin 6t - \sin 2t$$

$$= \sin 6t - \sin 2t$$

$$\mathcal{L}(f(t)) = \frac{6}{s^2+36} - \frac{2}{s^2+4}$$

b. $f(t) = \dfrac{e^{at} - \cos bt}{t}$ **(8)**

Answer:

Q. 8 (b) $f(t) = \dfrac{e^{at} - \cos bt}{t}$

Now, $\mathcal{L}(e^{at} - \cos bt) = \left(\dfrac{1}{s-a} - \dfrac{s}{s^2+b^2}\right)$

$$\therefore \mathcal{L}\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty \left(\frac{1}{(s-a)} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(s-a)^2 - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{(s-a)^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right]$$

$$= \frac{1}{2}\log\frac{(s^2+b^2)}{(s-a)^2}$$

Q.9 a. Find the inverse Laplace transform of $\dfrac{s-4}{4(s-3)^2+16}$ **(8)**

Answer:

Q.9 (a) $\mathcal{L}^{-1}\left\{\dfrac{s-4}{4(s-3)^2+16}\right\}$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{(s-3)-1}{(s-3)^2+2^2}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+2^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^2+2^2}\right\}$$

$$= \frac{1}{4}e^{3t}\cos 2t - \frac{1}{8}e^{3t}\sin 2t$$ (by first shifting theorem).

b. Apply convolution theorem to find $L^{-1} = \left\{\dfrac{s}{(s^2+1)(s^2+4)}\right\}$ **(8)**

Answer:

(b) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$

We know that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos x, \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2x$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \frac{1}{2}\int_0^t \sin 2x \cos(t-x)\,dx \quad \text{(by convolution thm)}$$

$$= \int_0^t \sin x \cos x \{\cos t \cos x + \sin t \sin x\}\,dx$$

$$= \int_0^t \left(\sin x \cos^2 x \cos t \cancel{dx} + \sin^2 x \cos x \sin t\right)dx$$

$$= \left[-\frac{\cos^3 x}{3}\cos t + \frac{\sin^3 x}{3}\cancel{\cos}\sin t\right]_0^t$$

$$= -\frac{\cos^4 t}{3} + \frac{\sin^4 t}{3} + \frac{1}{3}\cos t$$

$$= \frac{1}{3}(\sin^4 t - \cos^4 t) + \frac{1}{3}\cos t$$

$$= \frac{1}{3}(\sin^2 t - \cos^2 t) + \frac{1}{3}\cos t$$

TEXT BOOK

I. Engineering Mathematics – Babu Ram, Pearson Education Limited, 2012

II. Applied Mathematics for Polytechnic, H.K.Dass, 10th Edition, 2012, CBS Publishers & Distributors, New Delhi