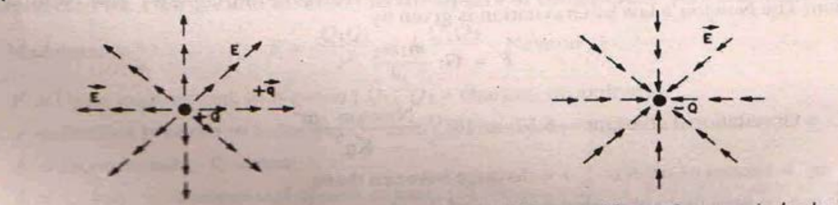
Q.2 a. Define the electric field intensity. Find the electric field intensity at a point P lying at a distance from an infinite straight uniform charged wire. (8)

Answer:

2.3. ELECTRIC FIELD INTENSITY

Electric field intensity or simply electric intensity or electric field is denoted by $\vec{E}$. If a small test (or probe) charge q is placed at any point near a second fix charge (Q), the probe charge q experiences a force. The magnitude and the direction of this will depend upon the location of the probe charge (q) w.r.t. fixed charge Q. About the charge Q, there is said to be an electric field of strength $\vec{E}$ and the magnitude of $\vec{E}$ at any point is measured as *force per unit charge* at that point. The direction of $\vec{E}$ is the direction of force on the positive test charge along the outward radial from the positive charge Q as illustrated in Fig. 2.2.



(a) Charge with positive numerical value. *(b) Charge with negative numerical value.*

Fig. 2.2. Fixed charge Q with vectors showing magnitude and direction of associated electric field.

Thus, the electric intensity $\vec{E}$ may be defined as *"The force per unit charge exerted on a test (or probe) charge in the field"*. It is sometimes also called as *"Electric field strength"* and its unit is volt / metre and may be found by applying Coulomb's Law, Eq. 2.5. The magnitnde of the force on the test charge q will be given by

$$F = \frac{Q \cdot q}{4 \pi \varepsilon r^2} \qquad \text{...(2.6)}$$

and the magnitude of the electric field intensity $\vec{E}$ due to fixed charge Q at test charge q is

$$E = \frac{F}{q} = \frac{Q \cdot q}{q \cdot 4 \pi \varepsilon r^2} \quad \text{or} \quad E = \frac{Q}{4 \pi \varepsilon r^2} \qquad \text{...(2.7)}$$

Thus from Eqn. 2.6 and 2.7, it is clear that the force on the test charge q is dependent upon the strength of the probe charge but Electric field intensity is not. Therefore, if the charge on the test charge is ...wed to approach zero, then the force per unit charge remains constant *i.e.* electric field due to fixed ... is considered to exist immaterial whether test charge q is there to detect its presence or not.

The direction and magnitude of electric field about a point charge ($q = 1$ for point charge) may ...ated by writing Eq. 2.7 in vector form *e.g.*

$$\ldots \frac{\ldots}{4 \pi \varepsilon_0 \, | r |}$$

ELECTRIC FIELD INTENSITY DUE TO INFINITELY LONG CHARGED WIRE [A LINE CHARGE]

Cartesian coordinate system

Consider the case of an infinitely long charged wire of negligible thickness as shown in Fig. (13). Let ρ_L be the density of charge per unit length. Our aim is to calculate the electric field at a point P, a distance r from the wire. For this purpose, we divide the wire into a number of infinitely small elements. Now, consider one small element of length dx at a distance x from O. The charge on this element is $dq = \rho_L \, dx$. The field dE at a point P due to this charge dq

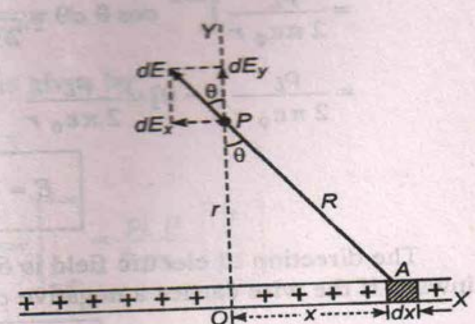


Fig. (13)

$$dE = \frac{1}{4\pi\varepsilon_0} \times \frac{dq}{R^2} = \frac{1}{4\pi\varepsilon_0} \times \frac{\rho_L\, dx}{R^2} \quad \text{(where } AP = R\text{)} \quad \ldots(1)$$

The x and y components of dE are

$$dE_x = -\,dE \sin\theta \quad \text{and} \quad dE_y = dE \cos\theta$$

Further consider an element of length dx at a distance x from O towards left. In this case, the components of dE along X and Y axis will be $dE \sin\theta$ and $dE \cos\theta$ respectively. Due to symmetry, the horizontal components of the field intensity will cancel each other and only vertical components remain. Hence, the total intensity of electric field at P will be $2\, dE \cos\theta$ in Y-direction.

$$\therefore \quad E = \int_0^{+\infty} 2\, dE \cos\theta \quad (\because \text{ now the wire extends from 0 to } +\infty)$$

or

$$E = \int_0^{\infty} \frac{1}{4\pi\varepsilon_0} \times \frac{\rho_L\, dx}{R^2} \times 2\cos\theta = \frac{\sigma}{2\pi\varepsilon_0} \int_0^{\infty} \frac{dx \cos\theta}{R^2} \quad \ldots(2)$$

From fig. $\frac{x}{r} = \tan\theta$

$$\therefore \quad \frac{dx}{r} = \sec^2\theta\, d\theta$$

or $dx = r \sec^2\theta\, d\theta$

and $R^2 = r^2 + x^2 = r^2 + r^2 \tan^2\theta$

when $x = 0,\ \theta = 0$ and when $x = \infty,\ \theta = \frac{\pi}{2}$

Substituting these values, we get

$$E = \frac{\rho_L}{2\pi\varepsilon_0} \int_0^{\pi/2} \frac{r\sec^2\theta\, d\theta \cdot \cos\theta}{(r^2 \tan^2\theta + r^2)} = \frac{\rho_L}{2\pi\varepsilon_0} \int_0^{\pi/2} \frac{r\sec^2\theta \cos\theta\, d\theta}{r^2\,(1 + \tan^2\theta)}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0} \int_0^{\pi/2} \frac{\sec^2\theta \cos\theta\, d\theta}{r \sec^2\theta} \quad (\because\ 1 + \tan^2\theta = \sec^2\theta)$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\, r} \int_0^{\pi/2} \cos\theta\, d\theta = \frac{\rho_L}{2\pi\varepsilon_0\, r} \left[\sin\theta\right]_0^{\pi/2}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\, r} [1 - 0] = \frac{\rho_L}{2\pi\varepsilon_0\, r}$$

$$\therefore \quad \boxed{E = \frac{\rho_L}{2\pi\varepsilon_0\, r}} \quad \ldots(3)$$

The direction of electric field is outward if the wire carries a positive charge and inward if the wire carries a negative charge.

In vector form

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\, r}\, \hat{\mathbf{a}}_r} \quad \ldots(4)$$

b. State and prove divergence theorem. (8)

Answer:

1.15. INTEGRAL THEOREMS

There are two most important types of integral theorems which will be of our interest.

(i) Divergence Theorem or Gauss's Theorem and (ii) Stroke's Theorem

Since, both theorems are of great use and hence will be discussed in detail.

1.15.1. Gauss's Divergence Theorem. This is an important theorem associated with the names of Green and Gauss and is popularly known as the Divergence Theorem. It permits us to express certain integrals by means of surface integrals. The divergence theorem relates to a closed volume V in space and the surface s that bounds it. In this it is assumed that there exists a vector function of position $\vec{A}$.

Statement. Gauss's divergence theorem states that *"the volume integral of the divergence of a vector field A taken over any volume V is equal to the surface integral of A taken over the closed surface surrounding the volume V".*

Mathematically, for any vector field $\vec{A}$

$$\iiint_V (\nabla \cdot \vec{A})\, d\vec{V} = \iint_S \vec{A} \cdot d\vec{s} = \iint_S \vec{A} \cdot \vec{e}\, d\vec{s} \qquad ...(1.52)$$

where $\vec{e}$ = unit vector outward normal to S and ds = an element of area on surface S. It may be noted here that triple integrals on the L.H.S. of equation 1.52 are because of the fact that the volume is a three dimensional quantity where as double integrals on the R.H.S. are used because the surface is two dimensional quantity. Since the theorem uses divergence of a vector function, hence the name divergence theorem.

Proof. For proof of the theorem L.H.S. of equation 1.52 will be proved equal to R.H.S. From definition, it is known that

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{also } dV = dx\, dy\, dz$$

Therefore, L.H.S. of Eqn. 1.52 can be written as

$$\iiint_V (\nabla \cdot \vec{A})\, dV = \iiint_V \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} dx\, dy\, dz$$

$$= \iiint_V \left(\frac{\partial A_x}{\partial x}\, dx\, dy\, dz \right) + \iiint_V \left(\frac{\partial A_y}{\partial y}\, dx\, dy\, dz \right) + \iiint_V \left(\frac{\partial A_z}{\partial z}\, dx\, dy\, dz \right) \qquad ...(1.53)$$

Consider now the first integral on the R.H.S. of Eqn. 1.53 *i.e.*

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$$\iiint_V \frac{\partial A_x}{\partial x}\, dx\, dy\, dz \qquad ...[1.53\ (a)]$$

Let us now consider the Fig. 1.20, in which a strip of cross-section $dydz$ extending from Q_1 to Q_2.

Integrating eqn. 1.53 (a) w.r.t. x *i.e.* along the shown strip (Fig. 1.20) of cross-section $dydz$ which extends from Q_1 to Q_2.

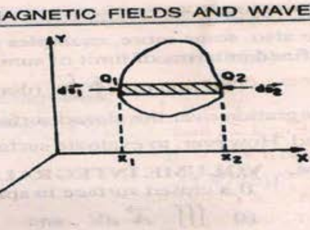

Fig. 1.20. Proof of divergence theorem.

$$\therefore \quad \int_{Q_1 = x_1}^{Q_2 = x_2} \frac{\partial A_x}{\partial x}\, dx = \left[A_x \right]_{x_1}^{x_2}$$

or $$\int_{x_1}^{x_2} \frac{\partial A_x}{\partial x} = (A_{x_2} - A_{x_1}) \qquad ...[1.53\ (b)]$$

$$\therefore \quad \iiint_V \frac{\partial A_x}{\partial x}\, dx\, dy\, dz = \iint_S \left[\int \frac{\partial A_x}{\partial x}\, dx \right] dy\, dz = \iint_S [A_{x_2} - A_{x_1}]\, dy\, dz$$

or $$\iiint_V \frac{\partial A_x}{\partial x}\, dx\, dy\, dz = \iint_S A_x ds_x \qquad ...[1.54\ (a)]$$

Surface integral on R.H.S. of Eqn. 1.54 (*a*) is evaluated over the whole surface. Here

At Q_2, $dy\, dz = ds_x$ and at Q_1, $dy\, dz = -ds_x$

Since the direction ds is along outward normal and hence x components of ds at Q_2 is ds_x and at Q_1 is $-ds_x$. Negative sign in later because its direction is opposite to that at Q_2.

Similarly, by considering the remaining two integrals in the R.H.S. of Eqn. 1.53, it can be seen that

$$\iiint_V \frac{\partial A_y}{\partial y}\, dx\, dy\, dz = \iint_S A_y\, ds_y \qquad ...[1.54\ (b)]$$

and $$\iiint_V \frac{\partial A_z}{\partial z}\, dx\, dy\, dz = \iint_S A_z\, ds_z \qquad ...[1.54\ (c)]$$

Now adding Eqn. 1.54 (*a, b, c*) we get

$$\iiint_V \frac{\partial A_x}{\partial x}\, dy\, dx\, dz + \iiint_V \frac{\partial A_y}{\partial y}\, dx\, dy\, dz + \iiint_V \frac{\partial A_z}{\partial z}\, dx\, dy\, dz$$

$$= \iint_S \left[A_x\, ds_x + A_y\, ds_y + A_z\, ds_z \right] = \iint_S [\vec{A} \cdot d\vec{s}] = \iint_S [\vec{A} \cdot \vec{e}\, d\vec{s}]$$

[From Eqn. 1.50 (a) $d\vec{s} = \vec{e}\, d\vec{s}$]

or $$\iiint_V \nabla \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot d\vec{s} \qquad \text{[From Eqn. 1.53]}$$

Thus divergence theorem is proved.

Q.3 a. State and explain the boundary condition at the interface of two dielectrics in an electrostatic field. (8)

Answer:

3.23. BOUNDARY CONDITIONS AT THE INTERFACE OF TWO DIELECTRICS

For solving any electrostatic problem involving more than one dielectric medium, it becomes necessary to know what happens at the boundary surfaces separating the different materials. It is rather necessary to relate the inaccessible (polarization) charges to the accessible (true) charges or to the fields produced by the latter. Such relationships which link the inaccessible charge source to the external fields which produce them are called the **constitutive equations.** For instance eqn. 3.95, although sometimes eqn. 3.96 is also called constitutive equations. These equations depend upon the properties of the material to which they apply. Equation 3.96 is restricted to linear, isotropic materials but the material need not be homogeneous *i.e.* ε may be a function of position. One very common case, where non-homogeneity occurs, is the dielectric constant ε varies discontinuously between two different homogeneous media and hence the way in which $\vec{D}$ and $\vec{E}$ behave in crossing the boundary between two dielectrics is of great interest to be discussed. Now the conditions (*i*) Normal components of $\vec{D}$ and $\vec{E}$, (*ii*) Conditions on Tangential components on $\vec{D}$ and $\vec{E}$ and (*iii*) Refraction of the lines of forces of $\vec{E}$ across a boundary will be considered.

3.23.1. Normal Components of $\vec{D}$ and $\vec{E}$. Let us imagine a disc enclosing a part of the boundary surface between two media and whose axis is normal to the boundary as shown in Fig. 3.15. Further, let ε_1 and ε_2 be the relative permitivities of the two media. It is assumed that the thickness of the disc is very small. The only contribution to the outward flux from the disc comes from its flat surfaces. Then by Gauss's law

$$\nabla \cdot \vec{D} = \rho.$$

Therefore,

$$\int_v \nabla \cdot \vec{D}\, dv = \int_v \rho\, dv \qquad \text{...[3.97 (b)]}$$

where ρ is the density of free charges. As there is no free charges at the surface, therefore,

$$\int_v \nabla \cdot \vec{D}\, dv = 0. \qquad \text{...(3.98)}$$

By divergence theorem, it is possible to change volume integral into surface integral as

$$\int_v \nabla \cdot \vec{D}\, dv = \int_s \vec{D} \cdot \vec{a}_n\, ds = 0 \qquad \text{...(3.99)}$$

or

$$\int_s \vec{D} \cdot \vec{a}_n\, ds = \vec{D}_1 \cdot \vec{a}_{n1}\, ds + \vec{D}_2 \cdot \vec{a}_{n2}\, ds = 0$$

where $\vec{D}_1$ and $\vec{D}_2$ are the values of $\vec{D}$ in the two media and $\vec{a}_{n1}$ and $\vec{a}_{n2}$ are the unit vectors. Since

$$\vec{a}_{n1} + \vec{a}_{n2} = 0$$

or

$$\vec{a}_{n1} = -\vec{a}_{n2}. \qquad \text{...(3.100)}$$

Hence eqn. 3.99 reduces to

$$\vec{D}_1 \cdot \vec{a}_{n1}\, ds - \vec{D}_2 \cdot \vec{a}_{n2}\, ds = 0 \qquad \text{...[3.101 (a)]}$$

or

$$(\vec{D}_1 \cdot \vec{a}_{n1} - \vec{D}_2\, \vec{a}_{n2})\, ds = 0 \qquad \text{...[3.101 (b)]}$$

or

$$\boxed{\vec{D}_1 \cdot \vec{a}_{n1} = \vec{D}_2 \cdot \vec{a}_{n2}.} \qquad \text{...(3.102)}$$

Hence, normal component of $\vec{D}$ in the two media are equal *i.e.* **$\vec{D}$ is continuous.**

Further, normal component of $\vec{E}$, on other hand, is discontinuous. This is obvious from Eqn. 3.96 (*b*)

i.e.
$$\vec{D} = \varepsilon \vec{E}.$$

Hence from Eqn. 3.102, we get $\varepsilon_1 \vec{E}_1 \cdot \vec{a}_{n1} = \varepsilon_2 E_2 \cdot \vec{a}_{n2}$

or
$$\boxed{\frac{\vec{E}_1 \cdot \vec{a}_{n1}}{\vec{E}_2 \cdot \vec{a}_{n2}} = \frac{\varepsilon_2}{\varepsilon_1}.} \quad ...(3.103)$$

In a simpler notations, if E_{1n} and E_{2n} are the normal component of $\vec{E}_1$ and $\vec{E}_2$, then Eqn. 3.103 reduces to

$$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}.} \quad ...(3.104)$$

Fig. 3.16 illustrated a very small element of the interface between dielectrics 1 and 2 whose permittivities are ε_1 and ε_2 respectively. As the element of surface is of differential extent, it is considered to be plane. The elemental shaped surface with its broad face is parallel to the interface and hence one surface is in region 1 and the other in region 2. The area of the broad face is supposed to be *ds* and the thickness is *h*. If Gauss's law is applied to this elemental volume as in Fig. 3.16, then by using Eqn.

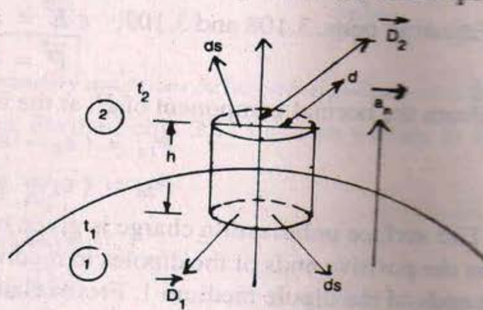

Fig. 3.16. Normal component of $\vec{D}$, determination thereoff.

$$\int_s \vec{D} \cdot d\vec{s} = \int_v \rho \, dv = Q \quad ...(3.105)$$

Q being total charge within the volume we have from Eqns. 3.101 or 3.105

$$\vec{a}_n \cdot \vec{D}_1 \, ds - \vec{a}_n \cdot \vec{D}_2 \, ds = \rho_s \, ds$$

or
$$\boxed{(\vec{a}_n \cdot \vec{D}_1) \, ds - (\vec{a}_n \cdot \vec{D}_2) \, ds = \rho_s \, ds} \quad ...(3.106)$$

where $\vec{a}_n$ = unit vector normal to interface.

For a simplest case of surface charge density ρ_s in free space medium

$$\vec{D}_1 = \varepsilon_0 \vec{E}_1 \quad \text{and} \quad \vec{D}_2 = \varepsilon_0 \vec{E}_2.$$

Then Eqn. 3.106 becomes

$$\vec{a}_n \cdot \varepsilon_0 \vec{E}_1 \, ds - \vec{a}_n \cdot \varepsilon_0 \vec{E}_2 \, ds = \rho_s \, ds$$

or
$$\vec{a}_n \cdot (\vec{E}_1 - \vec{E}_2) \, \varepsilon_0 \, ds = \rho_s \, ds$$

or
$$\boxed{\vec{a}_n \cdot (\vec{E}_1 - \vec{E}_2) = \rho_s / \varepsilon_0} \quad ...(3.107)$$

where ρ_s is the true charge density on the interface. It may be noted that in Eqn. 3.107, the outflux of $\vec{D}$ through the sides have not been included as this flow can be made negligible by letting $h \to 0$. While the terms, in Eqn. 3.107, due to top and bottom of the elemental volume can remain unaffected.

Hence by Eqn. 3.107, shows that **normal component of the electric field $\vec{E}$ is discontinuous through a charged surface and the magnitude of the discontinuity is given by, ρ_s/ε_0 (Eqn. 3.107).** For a dielectric interface ρ_s is normally equal to zero *i.e.* $\rho_s = 0$ unless a free surface charge is actually placed the interface. Hence by putting $\rho_s = 0$, eqn. 3.107 reduces to eqn. 3.102 which shows that **the normal component of $\vec{D}$ across a dielectric boundary containing no free charge is continuous.**

The discontinuity in $\vec{a}_n \cdot \vec{E}$ can be explained as below. The field $\vec{E}$ arises from the total effective charge, which consists of the true volume charge density ρ_v, the polarization volume charge density $\rho_p = -\nabla \cdot \vec{P}$ and the **polarization surface charge density** ρ_{sp}.

At the surface of a dielectric, the normal component of $\vec{E}$ is discontinuous by an amount equal to ρ_{sp}/ε_0 vide eqn. 3.107. Just as it would be if we consider a surface layer of true charge equal to ρ_{sp} - Eqn. 3.104 is readily shown to verify this result. From eqn.

$$D = \varepsilon_0 E + \vec{P} \quad \text{...(3.108)}$$

and

$$D = \varepsilon_0 E + \varepsilon_0 \chi_e E = \varepsilon_0 (1 + \chi_e) E$$

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}. \quad \text{...(3.109)}$$

Equating eqns. 3.108 and 3.109, $\varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$

or

$$\boxed{\vec{P} = (\varepsilon - \varepsilon_0) \vec{E}.} \quad \text{...(3.110)}$$

From the normal component of $\vec{P}$ at the interface is seen to be given by

$$P_{1n} = (\varepsilon_1 - \varepsilon_0) E_{1n} \text{ medium–1} \quad \text{...[3.111 (a)]}$$

$$P_{2n} = (\varepsilon_2 - \varepsilon_0) E_{2n} \text{ medium–2} \quad \text{...[3.111 (b)]}$$

The surface polarization charge is given by $P_{2n} - P_{in} P_{1n}$ because this represents the amount of charge on the positive ends of the dipoles in medium 2 that is not cancelled by the opposite charge on the negative ends of the dipole medium 1. From relation 3.111, we see that

$$P_{2n} - P_{in} = \rho_{sp}$$

or

$$(\varepsilon_2 - \varepsilon_0) E_{2n} - (\varepsilon_1 - \varepsilon_0) E_{1n} = \rho_{sp}$$

or

$$\varepsilon_2 E_{2n} - \varepsilon_0 E_{2n} - \varepsilon_1 E_{in} + \varepsilon_0 E_{in} = \rho_{sp}$$

or

$$\varepsilon_0 (E_{1n} - E_{2n}) + \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_{sp}. \quad \text{...(3.112)}$$

It is thus seen that a discontinuity of E_n by an amount ρ_{sp}/ε_0 corresponds to the requirement that

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}.$$

In otherwords Eqns. 3.103 and 3.104 are consistent with necessity that E_n be discontinuous by an amount ρ_{sp}/ε_0.

3.23.2. Tangential Components of $\vec{D}$ and $\vec{E}$. The tangential component of the Electric field $\vec{E}$ is continuous across the boundary between the two dielectric media or Tangential component of $\vec{D}$ is discontinuous across the boundary between dielectric media.

Let us now derive the boundary conditions on the tangential components of $\vec{D}$ and $\vec{E}$. For this purpose consider a cross-section normal to the interface as shown in Fig. 3.17 which separates two media of different permittivities.

Then a small closed rectangular path *abcda* of small length Δl and width Δh is taken so that the two opposite sides of longer dimensions lie in the two dielectric media.

Now the work done by the electric field $\vec{E}$ around the closed path *abcda* is

$$\int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0. \quad \text{...(3.113)}$$

Because the line integral of any electrostatic field around a closed path is zero. If $\Delta h \to 0$ keeping Δl fixed, then without affecting the two other integrals of Eqn. 3.113, we have

$$\int_{b \atop \Delta h \to 0}^c \vec{E} \cdot d\vec{l} + \int_{d \atop \Delta h \to 0}^a \vec{E} \cdot d\vec{l} = 0. \quad \text{...(3.114)}$$

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Let E_{1t} = Tangential component of $\vec{E}$ in the region of dielectric medium 1, and

E_{2t} = Tangential component of $\vec{E}$ in the region of the dielectric medium 2.

Then, with the choice of direction given in Fig. 3.17, we get from Eqns. 3.113 and 3.114

$$E_{1t}\,\Delta l - E_{2t}\,\Delta l = 0$$

or

$$E_{1t} = E_{2t}. \quad ...(3.115)$$

This means that **tangential component of the electric field is continuous across the boundary between two dielectric media**. In other words tangential components of Electric field are same on both sides of a boundary between two dielectrics. Further, eqn. 3.115 can be written in vector form as

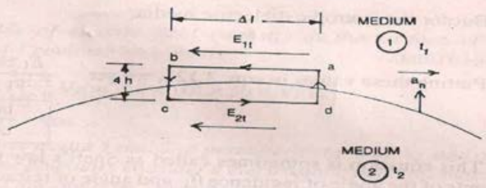

Fig. 3.17. Boundary conditions on tangential component of $\vec{E}$.

$$\vec{a}_n \times \vec{E}_1 = \vec{a}_n \times \vec{E}_2. \quad ...(3.116)$$

Thus, for the tangential components of $\vec{D}$, we must have

$$E_1 = \frac{D_{1t}}{\varepsilon_1} \text{ and } E_2 = \frac{D_{2t}}{\varepsilon_2}.$$

$$\therefore \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \quad \text{or} \quad \boxed{\frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}} \quad ...(3.117)$$

This shows that **the tangential component of flux density $\vec{D}$ is discontinuous.**

In case, the medium 2 is a conductor, then the field E_{2t} in medium 2 must be zero under static condition and then eqn. 3.115 reduces to

$$\boxed{E_{1t} = 0.} \quad ...(3.118)$$

Thus, according to eqn. 3.118, **the tangential component of electric field at a dielectric conductor boundary is zero.**

b. Drive the equation for the energy density in an electrostatics. (8)

Answer:

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or

$$\boxed{\frac{C}{l} = \frac{\pi\varepsilon}{\log_e\left[\left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]} = \frac{\pi\varepsilon}{\cosh^{-1}\left(\frac{D}{d}\right)}}$$

$$\boxed{\frac{C}{l} \simeq \frac{\pi\varepsilon}{\log_e\left[\frac{D}{R}\right]} \text{ Farad/metre}}$$

where R = Radius of conductor

D = Half centre to centre spacing

$\varepsilon = \varepsilon_o\,\varepsilon_r$

and ε_r = Relative permittivity of the medium surrounding the conductor...

2.43. ELECTROSTATIC ENERGY

When a capacitor is charged, there exists a potential difference *i.e.* voltage V between ... and electrostatic energy is stored. This energy can be converted into heat by discharging the capacitor through a resistance. The amount of electrostatic energy stored can be found by calculating the work done in charging the capacitor and potential is defined as *the work done per unit charge*. The work done in moving a small charge dQ against a potential difference V is VdQ *i.e.*

Work done $\quad dW = VdQ \quad$ or $\quad dW = \frac{Q}{C}\,dQ \quad$ But $V = \frac{Q}{C}$

If the capacitor is initially uncharged and the process of charging continued till a charge Q is achieved, then the total work done is

$$\int dW = \int_0^Q \frac{Q}{C}\,dQ$$

or

$$W = \frac{1}{C}\int_0^Q Q\,dQ = \frac{1}{C}\left[\frac{Q^2}{2}\right]_0^Q = \frac{1}{C}\cdot\frac{Q^2}{2} = \frac{Q^2}{2C} = \frac{V^2C^2}{2C} = \frac{V^2C}{2}$$

$$= \left(\frac{Q}{C}\right)\cdot\frac{Q}{2} = \frac{VQ}{2}$$

Hence, the total energy stored by a charged capacitor

$$\boxed{W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}V^2C = \frac{1}{2}VQ} \quad ...(2.234)$$

where Q = charge on one conductor, in coulomb.

C = capacitance, in Farad.

V = potential difference, in volts.

W = energy, in joules.

2.44. TOTAL ENERGY DENSITY IN A STATIC ELECTRIC FIELD

When a parallel plate capacitor is charged to a potential difference of V between plates, the energy stored is

$$W = \frac{1}{2}CV^2 = \frac{1}{2}VQ$$

Now question arises, which part of the capacitor is this energy stored ? The reply is that the energy is stored in the electric field between the plates. Let us see this.

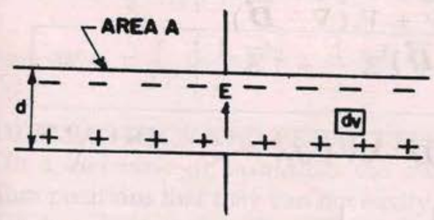

Fig. 2.55. Parallel plate capacitors and energy stored in the electric field.

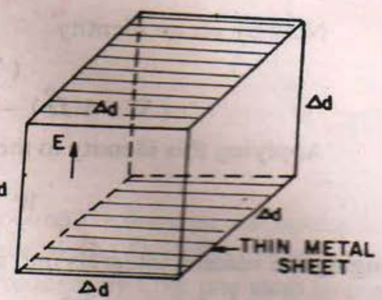

Fig. 2.56. Small cubical volume Δ V.

Consider the small cubical volume ΔV between the plates (Fig. 2.55). This volume is shown exaggerated in Fig. 2.56. The length of each side is Δd and top and bottom faces (of area Δd^2) are parallel to the capacitor plates (normal to the field $\vec{E}$). If the thin sheets of metal foil are placed coincident with the top and bottom faces of the volume, the field will be undisturbed provided the sheets are sufficiently thin. The volume ΔV now constitutes a small capacitor of capacitance

$$\Delta C = \varepsilon \cdot \frac{(\Delta d)^2}{\Delta d} \quad \text{or} \quad \Delta C = \varepsilon \cdot \Delta d$$

The potential difference ΔV of the thin sheets is $\Delta V = E\,\Delta d$

Now the energy ΔW stored in the volume ΔV is

$$\Delta W = \tfrac{1}{2}\,\Delta C \cdot (\Delta V)^2 = \tfrac{1}{2}\,(\varepsilon \cdot \Delta d)\,(E \cdot \Delta d)^2 = \tfrac{1}{2}\,\varepsilon\,E^2 \cdot (\Delta d)^3$$

$$\Delta W = \tfrac{1}{2}\,\varepsilon\,E^2 \cdot (\Delta V)$$

or

$$\frac{\Delta W}{\Delta V} = \frac{1}{2}\,\varepsilon\,E^2 \qquad \text{...(2.234)}$$

Now on taking the limit of the ratio as ΔV approaches zero, we obtain the energy per volume or energy density W at the point around which the volume shrinks to zero *i.e.*

$$w = \lim_{\Delta V \to 0} \frac{\Delta W}{\Delta V} = \frac{1}{2}\,\varepsilon\,E^2 \ \ \text{J/m}^3 \qquad \text{...(2.235)}$$

The total energy W stored by the capacitor is given by the integral of energy density W over the entire region in which the electric field $\vec{E}$ has a value

$$\boxed{W = \int_V w\,dv = \frac{1}{2}\int_V \varepsilon\,E^2\,dv}$$

$$W = \frac{1}{2}\int \varepsilon\,E \cdot E\,dv \qquad \because D = \varepsilon E$$

$$\boxed{W = \frac{1}{2}\int \vec{D} \cdot \vec{E}\,dv} \qquad \text{...(2.236)}$$

The latter can be deduced as follows : From eqn. 2.234 $\quad W = \tfrac{1}{2}\,VQ$

If the charge is distributed throughout the volume, the eqn becomes $W = \tfrac{1}{2}\int_V \rho_v\,V\,dv$

But by eqn. 2.73 $\quad \nabla \cdot \vec{D} = \rho$

$$\therefore \quad W = \frac{1}{2} \int (\nabla \cdot \vec{D})\, V\, dv$$

Now by vector identity

$$(\nabla \cdot V\vec{D}) = \vec{D} \cdot \nabla V + V\,(\nabla \cdot \vec{D})$$

or

$$(\nabla \cdot V\vec{D}) - \vec{D} \cdot \nabla V = V(\nabla \cdot \vec{D})$$

Applying this identity to the integrand gives

$$W = \frac{1}{2} \int_V [\nabla \cdot V\vec{D} - \vec{D} \cdot \nabla V]\, dv$$

changing the volume integrals into surface integral, we get

$$W = \frac{1}{2} \int_V \nabla \cdot V\vec{D}\, dv - \frac{1}{2} \int_V \vec{D} \cdot \nabla V\, dv = \frac{1}{2} \int_S V\vec{D}\, d\vec{s} - \frac{1}{2} \int_V \vec{D} \cdot \nabla V\, dv$$

As the enclosing sphere becomes very large, the enclosed volume charge looks like a point charge. Thus, at the surface $\vec{D}$ varies inversely as square of distance $\left(1/r^2\right)$ and potential (V) varies as $1/r$, so the integrand is decreasing as $1/r^3$. Since the surface area increases as r^2 it follows that $\lim_{r \to \infty} \int_S V\vec{D}\, d\vec{s} = 0$

$$\therefore \quad W = -\frac{1}{2} \int_V \vec{D} \cdot \nabla V\, dv \qquad \because \nabla V = -\vec{E}$$

$$\boxed{W = +\frac{1}{2} \int_V \vec{D} \cdot \vec{E}\, d\vec{v}} \qquad \text{Since } \vec{D} = \varepsilon \vec{E}$$

$$\boxed{W = \frac{1}{2} \int \varepsilon E^2\, dv = \frac{1}{2} \int \frac{D^2}{\varepsilon}\, dv}$$

Assuming that the field is uniform between the plates and that there is no frining of the field at the edges of the capacitor,

$$W = \tfrac{1}{2}\, \varepsilon E^2 \int dv = \tfrac{1}{2}\, \varepsilon\, E^2 \cdot Ad = \tfrac{1}{2}\, \varepsilon\, \vec{E} \cdot AEd = \tfrac{1}{2}\, D \cdot AEd \qquad \because Q = DA,\ V = Ed$$

$$= \tfrac{1}{2}\, DA \cdot Ed = \tfrac{1}{2}\, Q \cdot V \quad \text{Joules}$$

where A = Area of one capacitor plate, in m^2.

d = spacing between capacitor plates, in m.

Obviously this result which has been obtained by integrating the energy density throughout the volume between capacitor plates, is identical with that already obtained in previous article.

Further, electromagnetic field theory makes it easy to believe that the energy of an electric field or charge distribution is stored in the field itself as can be seen from eqn. 2.236.

$$W = \tfrac{1}{2} \int_V \vec{D} \cdot \vec{E}\, dv$$

or in differential form.

$$dw = \frac{1}{2}\, \vec{D} \cdot \vec{E}\, dv \quad \text{or} \quad \frac{dw}{dv} = \frac{1}{2}\, \vec{D} \cdot \vec{E} \quad \text{Joules/m}^3$$

obviously, this has the dimensions of an energy density or J/m^3.

If there are two electric fields $\vec{E}_1$ and $\vec{E}_2$, then total energy density stored is given by eqn. 2.239

$$W = \frac{1}{2}\left[\int_V \varepsilon\left\{\vec{E}_1 + \vec{E}_2\right\}^2 dv\right] = \frac{1}{2}\,\varepsilon\left[\int_V \left\{\vec{E}_1^{\,2} + \vec{E}_2^{\,2} + 2\,\vec{E}_1\,\vec{E}_2\right\} dv\right]$$

$$= \frac{1}{2}\,\varepsilon \int_V \left\{\vec{E}_1^2 + \vec{E}_2^2 + 2\vec{E}_1\,\vec{E}_2\right\} dv$$

$$\boxed{W = \varepsilon \int_V \left\{\frac{1}{2}\,\vec{E}_1^2 + \frac{1}{2}\,\vec{E}_2^2 + \vec{E}_1\,\vec{E}_2\right\} dv} \qquad ...(2.241)$$

Q.4 a. Derive Poisson's and Laplace's equation. (8)

Answer:

2.29. POISSON'S EQUATION AND LAPLACE'S EQUATION

Besides divergence operator, there is another Laplacian (Laplah-ci-an) operator. Eqn. 2.71 is a relation between the flux density $\vec{D}$ and the charge density ρ that exist in the region.

Thus $\nabla \cdot \vec{D} = \rho$

But $\vec{D} = \varepsilon \vec{E}$

$\therefore$ $\nabla \cdot (\varepsilon \vec{E}) = \rho$

If the region is homogeneous and isotropic, the dielectric const or permittivity ε will be scalar quantity, and hence.

$$\varepsilon\, \nabla \cdot \vec{E} = \rho \qquad \text{But } \vec{E} = -\nabla V$$

$$\therefore \quad -\varepsilon\, \nabla \cdot (\nabla V) = \rho$$

or

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}} \qquad ...(2.173)$$

This Eqn. is known as **Poisson's equation and is useful in vacuum tubes and gaseous discharge problems particularly.**

The divergence of a gradient (the double operator) is written as ∇^2 (del square) and is called as the *Laplacian operator.*

In free space when there is no charge (*i.e.* $\rho = 0$), above eqn. becomes

$$\boxed{\nabla^2 V = 0} \qquad ...(2.174)$$

This eqn. is known as *Laplace's equation.*

Expanding equation 2.174 in rectangular co-ordinate, we get,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(2.175)$$

Further when $\rho = 0$, then eqn. 2.74.

$$\nabla \cdot \vec{D} = 0$$

or

$$\nabla \cdot \varepsilon \vec{E} = 0$$

or

$$\nabla \cdot \vec{E} = 0 \qquad ...(2.176)$$

Laplace's eqn. is of great importance in electromagnetic theory. Eqn. 2.174 is special case of Poisson's eqn. for charge free regions but eqns. 2.175 and 2.176 are the alternative forms.

or in vector form

$$\vec{E} = E_r \, \vec{a_r}$$

or

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r} \vec{a_r} \ \text{V/m} \quad \text{...[2.86 (b)]}$$

It is thus seen that the result is independent of the radius of charged cylinder R and hence holds good for the rectilinear distribution of charges.

Now the potential difference between two points A and B at a distance "a" and "b" respectively from the centre of the charge wire having a charge ρ_l Coulomb/metre is given by

$$V_A - V_B = -\int_b^a \vec{E} \cdot d\vec{r} = -\int_b^a \vec{E}_r \cdot dr$$

$$V_{ab} = -\int_b^a \frac{\rho_l}{2\pi\varepsilon_0 r} \cdot dr = -\frac{\rho_l}{2\pi\varepsilon_0} \log_e \frac{a}{b}$$

$$V_{ab} = \frac{\rho_l}{2\pi\varepsilon_0} \log_e \frac{b}{a} \quad \text{...(2.87)}$$

Further, potential difference between any point on the charged wire and point B due to charge ρ_l on the wire (Fig. 2.31) we may have,

$$V_A - V_B = -\int_b^R \vec{E}_r \, d\vec{r}$$

or

$$V_{ab} = -\int_b^R \frac{\rho_l}{2\pi\varepsilon_0 r} dr = -\frac{\rho_l}{2\pi\varepsilon_0} \log_e \left(\frac{R}{b}\right)$$

$$V_{ab} = \frac{\rho_l}{2\pi\varepsilon_0} \log_e \left(\frac{b}{R}\right) \ \text{Volts} \quad \text{...(2.88))}$$

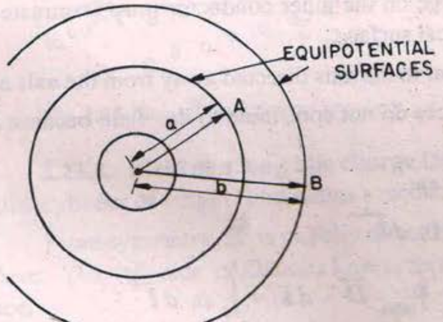

Fig. 2.31. Potential difference between two equipotential surfaces.

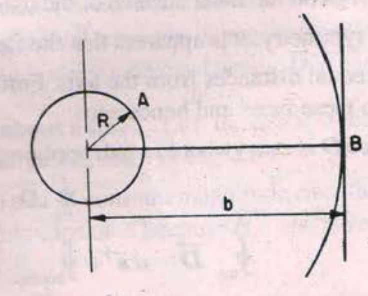

Fig. 2.32. Potential difference between two coaxial cylinder.

2.23. LAPLACE'S EQUATION

Electric field intensity $\vec{E}$ was determined in the beginning of the chapter by summation or integration of point charges, line charges, surface charges and volume charges. Subsequently, Gauss's Law was used to determine $\vec{D}$ which then gave $\vec{E}$ as $\vec{D} = \epsilon \vec{E}$. Although these two approaches are important and give valuable assistance in understanding the electromagnetic field theory, yet both methods tend to be impracticable as charge distributions are not usually known. Still another method of calculating $\vec{E}$ is by using the relation $\vec{E} = -\nabla V$ in which negative gradient of potential V is involved and this requires that the potential function throughout the region be known, which is generally not. Instead of above, Sometimes conducting materials in the form of planes, curved surfaces or lines are usually specified and the voltage on one is known w.r.t. some other reference, often one of the conductors. Laplace's equation then provides a powerful method whereby the potential function V can be calculated subject to the conditions on the bounding conductors.

Since the left side of Laplace's equation (eqn. 2.102) is the divergence of the gradient of V, these two operations can be used to reach at the form of the equation in a particular co-ordinate system. The Laplace's equations in three co-ordinate for a general vector field $\vec{A}$ and potential function V in all the three co-ordinates are given below.

(i) *Cartesian Co-ordinate. For potential function V and general vector field* $\vec{A}$

$$\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

or

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z$$

and

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right) \cdot (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)$$

$$\boxed{\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \qquad \because \vec{a}_x \cdot \vec{a}_x = 0 \text{ etc.}$$

and hence Laplace's equation.

$$\nabla(\nabla V) = \nabla^2 V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y} + \frac{\partial^2 V}{\partial z} = 0 \qquad \text{...(2.89)}$$

where $\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$ are unit vectors along three coordinate axes.

(ii) *Cylindrical Co-ordinates.*

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_r + \frac{1}{r}\frac{\partial V}{r\varphi}\vec{a}_\varphi + \frac{\partial V}{\partial z}\vec{a}_z$$

and general vector field $\vec{A}$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial Az}{\partial z}$$

and hence Laplaces equation is

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0} \qquad \text{...(2.90)}$$

(iii) *Spherical Co-ordinates.*

$$\nabla V = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\vec{a}_\varphi$$

for general vector field A

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

and Laplace's equation is, therefore

$$\boxed{\nabla^2\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 V}{\partial \varphi^2} = 0} \qquad \text{...(2.91)}$$

b. Derive Laplace's equation for parallel plate capacitor in rectangular coordinate and determine C there from. (8)

Answer:

values.

2.24. SOLUTION OF LAPLACE'S EQUATION IN RECTANGULAR COORDINATES

Several methods are available for solving the second order partial differential equation known as Laplace's equation. The first and simplest method is the direct integration method. The other method is the product method used in difficult problems. Still another method requires advanced mathematical knowledge. The direct integration method is applicable, however, only to problems of one dimensions or in which the potential field is a function of only one of the three co-ordinates.

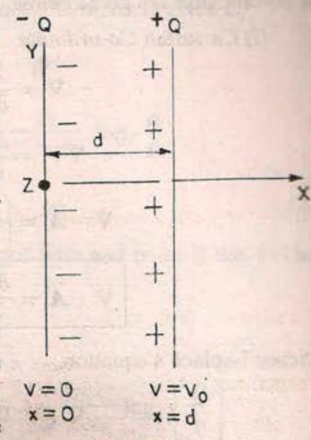

Fig. 2.33. Parallel plate capacitor.

2.24.1. Cartesian solution in one dimension (field between two parallel plates). Let us consider a parallel plate capacitor as shown in Fig. 2.33. Plate to the left is at zero potential and that at right at potential V_0. We will use the Laplace's equation to calculate the potential distribution between the plates.

Since there is no variation in y and z direction, the problem is one dimensional and Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{reduces to} \quad \frac{\partial^2 V}{\partial x^2} = 0. \qquad ...(2.92)$$

The partial derivative is also replaced by ordinary derivative as V is not a function of y and z

$$\therefore \qquad \frac{d^2V}{dx^2} = 0 \quad \text{or} \quad \int \frac{\partial^2 V}{\partial x^2} = \int 0$$

or

$$\frac{dV}{dx} = \text{Const.} = A \text{ (Say)} \qquad ...(2.93)$$

Again integrating

$$\int \frac{dV}{dx} = \int A \quad \text{or} \quad \int dV = \int A\, dx$$

or

$$V = Ax + B \qquad ...(2.94)$$

where A and B are constants of integrations which can be determined by boundary conditions.

Boundary conditions are

At $\quad x = 0, \quad V = 0 \qquad ...[2.95 (a)]$

$\quad x = d \quad V = V_o \qquad ...[2.95 (b)]$

Putting Eq. 2.95 (a) in Eq. 2.94 we get

$$B = 0 \text{ i.e. } V = A_x$$

and by putting eqn. 2.95 (b) into eqn. 2.94, we get

$$V_0 = A\,d \quad \text{or} \quad A = \frac{V_0}{d}$$

Hence introducing these into eqn. 2.96, we get

$$\boxed{V = \frac{V_0}{d}\,x \ \text{ Volts}} \qquad \text{...(2.96)}$$

The boundary conditions are at choice. In general, say

At $\quad x = d_1 \quad V = V_1$ and $x = d_2 \quad V = V_2$

Then $\quad V_1 = Ad_1 + B \quad$ and $\quad V_2 = Ad_2 + B$

From these by subtracting, we get

$$V_1 - V_2 = A\,(d_1 - d_2) \quad \text{or} \quad A = \frac{V_1 - V_2}{d_1 - d_2} \qquad \text{...[2.97 (a)]}$$

and also by multiplying eqns. by d_2 and d_1 respectively, and subtracting, we get

$$V_1\,d_2 = A\,d_1\,d_2 + Bd_2 \text{ and } V_2\,d_1 = A\,d_1\,d_2 + Bd_1$$

$$\therefore \quad V_1 d_2 - V_2 d_1 = B\,(d_2 - d_1)$$

or

$$B = \frac{V_1\,d_2 - V_2\,d_1}{d_2 - d_1} \qquad \text{...[2.97 (b)]}$$

and hence

$$\boxed{V = \left(\frac{V_1 - V_2}{d_1 - d_2}\right) x + \frac{V_1\,d_2 - V_2\,d_1}{d_2 - d_1}} \qquad \text{...(2.98)}$$

Since the capacitance is the ratio of charge to potential and since the potential has been known, charge on either plate is yet to be determined. This is done as follows :

(a) Calculate E from $E = -\nabla V$, if V is given ;

(b) Calculate $\vec{D}$ from $\vec{D} = \varepsilon\,\vec{E}$;

(c) Calculate $\vec{D}$ at either Capacitor plate, since $\vec{D} = \vec{D}_s = \vec{D}_n\,\vec{a}_n$;

(d) Remember $\rho_s = D_n$;

(e) Calculate Q by a surface integration over the capacitor plate *i.e.*

$$Q = \int_s \rho_s\,ds \qquad \text{...(2.99)}$$

$$\because \quad V = V_0\,\frac{x}{d} \qquad \therefore \ \frac{\partial V}{\partial x} = \frac{V_0}{d}$$

and

$$E = -\nabla V$$

$$\vec{E} = -\frac{\partial V}{\partial x}\,\vec{a}_x \qquad \text{...(2.100)}$$

or

$$\vec{E} = -\frac{V_0}{d}\,\vec{a}_x$$

$$\therefore \quad D = +\,\varepsilon\,E$$

$$\boxed{D = -\,\varepsilon\,\frac{V_0}{d}\,\vec{a}_x} \qquad \text{...[2.101 (a)]}$$

$$\therefore \quad D_s = [\,D\,]_{x=0} = -\,\varepsilon\,\frac{V_0}{d}\,\vec{a}_x \qquad \text{...[2.101 (b)]}$$

But $\quad \vec{D} = \vec{D}_s = D_n\,\vec{a}_n$

or $\quad D_n\,\vec{a}_n = -\,\frac{\varepsilon\,V_0}{d}\,\vec{a}_x \qquad \because \ \vec{a}_n = \vec{a}_x$

or $\quad D_n = -\,\frac{\varepsilon\,V_0}{d} = \rho_s \qquad \text{...[2.101 (c)]}$

Hence $\quad Q = \int_s \rho_s\,ds = -\int_s \frac{\varepsilon\,V_0}{d}\cdot ds = -\,\frac{\varepsilon\,V_0}{d}\int_s ds \qquad \because \ \int_s ds = A$

$$Q = -\,\frac{\varepsilon\,V_0\,A}{d}$$

or $\quad |\vec{Q}| = \left|\frac{-\,\varepsilon\,V_0\,A}{d}\right| = +\,\frac{\varepsilon\,V_0\,A}{d}$

Hence, $\quad C = \frac{Q}{V_0} = \frac{\varepsilon\,A}{d} \quad$ Farad $\qquad \text{...(2.102)}$

This shows the use of Laplace's equation involving minimum mathematics.

Q.5 a. Using Ampere's law, calculate the magnetic field intensity at a point due to line current placed along the z-axis extending from -∞ to ∞. (8)

Answer:

5.4 APPLICATIONS OF AMPERE'S LAW

5.4.1 MAGNETIC FIELD INTENSITY DUE TO INFINITELY LONG CONDUCTOR CARRYING A CURRENT *I* (A)

Consider an infinite conductor placed along the *Z*-axis as shown in fig. (19). The aim of this article is to derive an expression for **H** at a point distant ρ from origin. By symmetry, the magnetic field is directed in ϕ direction everywhere and is constant for fixed radius ρ. So, we construct a circle of radius ρ parallel to *X*-*Y* plane as shown in the figure by dotted curve.

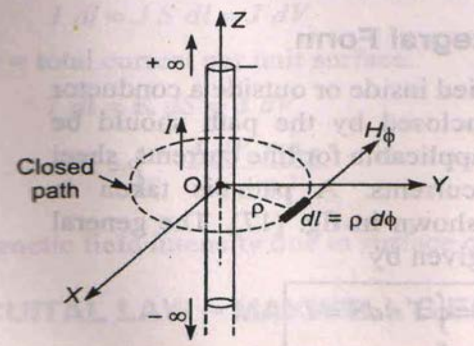

Fig. (19)

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Applying Ampere's circuital law to this closed circuit, we have

$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$

or $$H_\phi \oint dl = I$$

or $$H_\phi\,[2\pi\rho] = I$$

or $$H_\phi = \frac{I}{2\pi\rho}$$

Hence, the magnetic field intensity in vector form is

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\,\hat{\mathbf{a}}_\phi} \quad \text{A/m}$$

The magnetic flux density is given by

$$\mathbf{B} = \mu\,\mathbf{H} = \frac{\mu I}{2\pi\rho}\,\hat{\mathbf{a}}_\phi \text{ tesla.}$$

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$$\hat{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{z\,\hat{a}_z - r\,\hat{a}_r}{\sqrt{(z^2 + r^2)}} \quad \ldots(3)$$

According to Biot-Savart law, the magnetic field intensity at point P is given by

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \hat{a}_R}{4\pi R^2} \quad \ldots(4)$$

Substituting the values of $d\mathbf{l}$ and $\hat{a}_R$ from eqs. (1) and (3) in eq. (4), we get

$$d\mathbf{H} = \frac{I\, r\, d\phi\; \hat{a}_\phi}{4\pi (r^2 + z^2)} \times \frac{z\,\hat{a}_z - r\,\hat{a}_r}{\sqrt{(r^2 + z^2)}}$$

$$= \frac{I\, r\, d\phi\, [z\,(\hat{a}_\phi \times \hat{a}_z) - r\,(\hat{a}_\phi \times \hat{a}_r)]}{4\pi (r^2 + z^2)^{3/2}}$$

$$= \frac{I\, r\, z\, d\phi\; \hat{a}_r}{4\pi (r^2 + z^2)^{3/2}} + \frac{I\, r^2\, d\phi\; \hat{a}_z}{4\pi (r^2 + z^2)^{3/2}} \quad \ldots(5)$$

$$(\because\ \hat{a}_\phi \times \hat{a}_z = \hat{a}_r \quad \text{and} \quad \hat{a}_\phi \times \hat{a}_r = -\,\hat{a}_z)$$

In eq. (5) $\hat{a}_z$ is the vertical component while $\hat{a}_r$ is the horizontal component, when we take integration of eq. (5) to obtain total magnetic field intensity, all the horizontal components get cancelled. This is due to the fact that circular loop is placed in X-Y plane. Therefore,

$$\mathbf{H} = \int_{\phi=0}^{2\pi} \frac{I\, r^2\, d\phi\; \hat{a}_z}{4\pi (r^2 + z^2)^{3/2}}$$

or

$$\mathbf{H} = \frac{I\, r^2\, \hat{a}_z}{4\pi (r^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi = \frac{I\, r^2\, \hat{a}_z}{4\pi (r^2 + z^2)^{3/2}} \times 2\pi$$

$$\therefore \quad \boxed{\mathbf{H} = \frac{I\, r^2\, \hat{a}_z}{2\,(r^2 + z^2)^{3/2}}} \quad \text{weber/metre}^2 \text{ or tesla} \quad \ldots(6)$$

For example, consider a loop located on $x^2 + y^2 = 9$ and carries a current of 5 amp. The magnetic field $\mathbf{H}$ at (0, 0, 4) and (0, 0, − 4) will be

$$\mathbf{H} \text{ at } (0, 0, 4) = \frac{5\,(3)^2\,\hat{a}_z}{2\,[9 + 16]^{1/2}} = 0.18\,\hat{a}_z \text{ A/m} \quad (\because I = 5\text{ A}, \rho = 3 \text{ and } h = 4)$$

$$\mathbf{H} \text{ at } (0, 0, -4) = 0.18\,\hat{a}_z \text{ A/m}$$

Because when h is replaced by $-h$, the z component of $d\mathbf{H}$ remains the same while ρ component still adds up to zero due to axial symmetry).

b. Apply Biot-Savart's law to calculate magnetic field of a circular current carrying loop. (8)

Answer:

or $4\pi\rho$

5.2.3 MAGNETIC FIELD INTENSITY ON THE AXIS OF A CIRCULAR LOOP

As shown in fig. (5), consider a circular loop of radius r and carrying a current I. Conser a ponit P (0, 0, z) on Z-axis. The aim of this article is to find out the magnetic field intensity at this point.

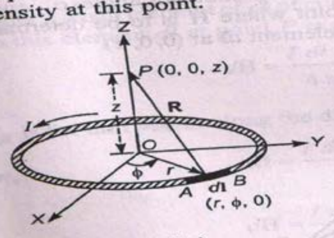


Fig. (5a)

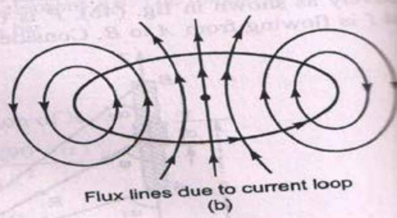


Fig. (5b)

Consider a small length dl of the circular loop. This length can be considered as a point. As the circular loop is placed in X-Y plane, hence in this plane $z = 0$. So, the coordinates of dl are $(r, \phi, 0)$.

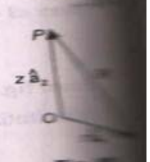
In cylindrical coordinate system, the expression for $d\mathbf{l}$ is given by

$$d\mathbf{l} = dr\;\hat{a}_r + r\, d\phi\;\hat{a}_\phi + dz\;\hat{a}_z$$

For a circular loop, r = constant, *i.e.*, $dr = 0$ and here $z = 0$.

$$\therefore \quad d\mathbf{l} = r\, d\phi\;\hat{a}_\phi$$

To find the value of $\mathbf{R}$, we consider fig. (6). Here $\hat{a}_r$ and $\hat{a}_z$ are unit vectors along r and Z directions respectively. From figure,



$$z\,\hat{a}_z = r\,\hat{a}_r + \mathbf{R}$$

or

$$\mathbf{R} = z\,\hat{a}_z - r\,\hat{a}_r$$

or

$$|\mathbf{R}| = R = \sqrt{(r^2 + z^2)} \quad \ldots(2)$$

Let $\hat{a}_R$ be the unit vector in the direction of $\mathbf{R}$. Then

Q.6 a. Find the force between an infinite straight line wire carrying a current I_1, and a square loop of side a with current I_2, the extended plane of loop containing the straight line wire. The shortest distance from the wire to the loop is d and the wire lies parallel to one side of the loop. (8)

Answer:

... inductance for 10 m long = $10 \times 0.22186 \times$ mH = 0.2218 mh **Ans.**

Example 4.20. *Find the force between an infinite straight line wire carrying a current I_1 and a square loop of side a with current I_2, the extended plane of loop containing the straight line wire. The shortest distance from the wire to the loop is d and wire lies parallel to one side of the loop.*

(AMIE, May 1974, Winter 1981)

Solution. Let C_2 be a square loop carrying current I_2 placed with nearest side at distance d from an infinite wire C_1 carrying current I_1. If the straight conductor carrying a current I_1 is in positive y-direction, the field produced in the loop is in the negative z-direction as shown by "cross" in the Fig. 4.59.

Now the flux linking C_2 due to current I_1 (of C_1) is given by Eqn. 4.42

$$\psi_m = \int_S \vec{B} \cdot d\vec{s}$$

and magnetic flux density at a distance d metres from the infinite straight wire carrying current I_1 is given by

$$B = \frac{\mu I_1}{2 \pi x} \text{ Wb/m}^2 \qquad \text{...[4.178 (a)]}$$

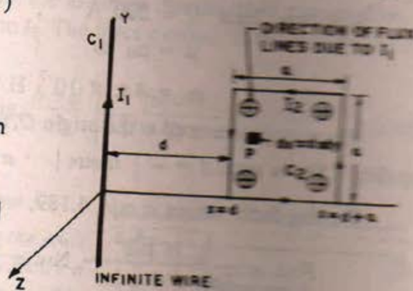


Fig. 4.59. Straight conductor and square loop.

Mutual Inductance is given by the flux linkage with the loop per unit current in the straight wire. If Φ is the flux linkage with the coil, and I_1 is the current in the wire, then mutual inductance

$$M = \frac{\Phi}{I_1} = \frac{N \psi_m}{I_1}$$

Since coil is single turn and hence $N = 1$ *i.e.* $\Phi = \psi_m = \int \vec{B} \cdot d\vec{s}$

where ds is a surface element around any point $P(x, y)$ in the loop plane surface and

$$ds = dx\, dy$$

Since flux density $\vec{B}$ is a vector normal to the plane of the coil, we can write

$$\Phi = \int_{C_1}\int_{C_2} \vec{B}\, d\vec{s} = \int_{y=0}^{y=a}\int_{x=a}^{x=d+a} \frac{\mu I_1}{2\pi x}\, dx\, dy = \frac{\mu I_1}{2\pi}\int_{x=d}^{x=d+a}\left[\frac{dx}{x}\right]\int_{y=0}^{y=a} dy$$

$$= \frac{\mu I_1}{2\pi}\int_{x=d}^{x=d+a}\frac{dx}{x}\Big[y\Big]_0^a = \frac{\mu I_1 a}{2\pi}\Big[\log_e x\Big]_{x=d}^{x=d+a} = \frac{\mu I_1 a}{2\pi}\left[\log_e \frac{d+a}{d}\right]$$

$$M = \frac{\Phi}{I_1} = \frac{\mu a}{2\pi}\log_e\left(\frac{d+a}{d}\right) \text{ H} \qquad \text{...[4.178 (b)]}$$

The force exerted by the field on C_2 in the direction of increasing d is given by

$$F = I_1 I_2 \frac{d}{dx}(M) \qquad \because x = d \text{ here}$$

$$= I_1 I_2 \frac{d}{d(d)}\left\{\frac{\mu a}{2\pi}\log_e \frac{d+a}{d}\right\} = \frac{I_1 I_2 \mu a}{2\pi} \cdot \frac{1}{\frac{d+a}{d}} \cdot \frac{d}{d(d)}\left(\frac{d+a}{d}\right)$$

$$F = \frac{I_1\, I_2\, \mu\, a\, d}{2\pi\,(d+a)}\,\frac{d}{d(d)}\left(1+\frac{a}{d}\right) = \frac{I_1\, I_2\, \mu\, a\, d}{2\pi\,(d+a)}\left(-\frac{a}{d^2}\right) \qquad \because \frac{d}{d(d)} = -\frac{a}{a^2}$$

$$\boxed{F = -\frac{I_1\, I_2\, \mu\, a^2}{2\pi\,(d+a)\,d}\ \text{Nw}} \qquad \text{...(4.179)}$$

This shows that the current loop is attracted towards the infinitely long straight wire.

b. Write note on. (8)

(i) Hysteresis loss

(ii) Retarded potential

Answer:

... the tips of these loops is the normal magnetisation curve.

HYSTERESIS LOSS

Magnetic hysteresis or simply hysteresis is defined as the lagging of magnetic flux density B behind the magnetic field intensity H and energy lost in carrying out the magnetization of a ferromagnetic material around the hysteresis loop is known as Hysteresis loss. It is that quantity of magnetic material due to which energy is dissipated in it on the reversal of its magnetism. The Hysteresis loss is caused by the work required to magnetise the magnetic material and area of the hysteresis loop represents the work done per unit volume in one cycle around the hysteresis loss. This amount of energy is dissipated as heat in the magnetic material and is called hysteresis loss.

Let us now calculate the hysteresis loss. Fig. 5.13 shows the hysteresis loop for a toroid shown in Fig. 5.14 with mean radius R, area of cross-section A, having N turn through which a current I flows. Let the initial state of magnetisation is represented by "a" on the hysteresis loop. A slight increase in current I (*i.e.* dI) there will be increase in the value of H by dH and in the value of B by dB. With this slight increase in the value of B, some voltage (given below), will be produced in the toroid winding

$$e = -N\,\frac{d\,\psi_m}{dt} = -N\left[\frac{d\,(BA)}{dt}\right] = -NA\,\frac{dB}{dt}. \qquad \text{...(5.15)}$$

In eqn. 5.15 dt is the time in which increase took place. The battery will have to do work against the voltage induced for increasing the current from I to $I + dI$. If this work done is dW, then

$$dW = -\,e\,I\,dt = +\,NA\,I\,dB. \qquad \text{...(5.16)}$$

Now according to Ampere's circuital law $H = \dfrac{NI}{l}$, where $l = 2\pi R$

$$H \cdot 2\pi R = NI. \qquad \text{...(5.17)}$$

Putting this in Eqn. 5.16, we get $dW = H \cdot 2\pi R \cdot A\,dB = VH\,dB$ ...(5.18)

where $V = 2\pi RA$ = Volume of the toroid.

Now in order to change the value of B upto B_1 along the path ab, the battery will have to do work say W_1, then

$$W_1 = \int_0^{B_1} dW = \int_0^{B_1} VH\,dB = V\int_0^{B_1} H\,dB = V\,(A_1 + A_2) \qquad \text{...(5.19)}$$

where A_1 and A_2 are the areas as indicated in the Fig. 5.13. Further to decrease the value of B from B_1 to B_2 along the path bc, work will be done against the battery say W_2, then

$$W_2 = V\int_{B_1}^{B_2} A\,dB = -\,V\,A_2. \qquad \text{...(5.20)}$$

This is because decreasing flux will produce a voltage which tend to maintain the current. Further, in order to reduce the value of B to O along the path cd, work have to be done by the battery as the direction of current will change while voltage induced will remain in the same direction as in Eqn. 5.20. Let this work be W_3 and is given by

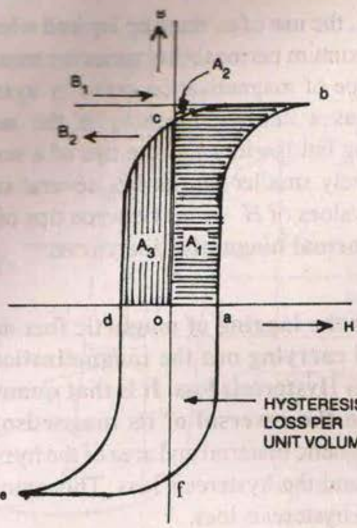
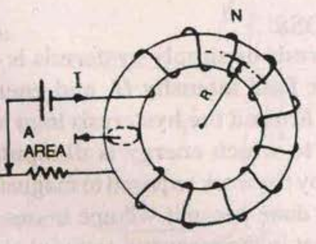


Fig. 5.13.



Toroid with area A Current I, Turn N
Fig. 5.14.

$$W_3 = V A_3 \qquad \text{...(5.21)}$$

where the area A_3 is indicated in Fig. 5.13.

Now for magnetising the material along the paths d to e and back to a, the work done will be the same as that required to magnetise along the upper half of the loop. Hence, the total work done by the battery for magnetising material around the loop in one complete cycle is given by twice *i.e.*

$$W = 2\,V(W_1 + W_2 + W_3) = 2\,V(A_1 + A_2 - A_2 + A_3) = 2\,V(A_1 + A_3)$$
$$= V \cdot 2(A_1 + A_3) = V \cdot A$$

or $$\frac{W}{V} = A \qquad \text{...(5.22)}$$

where $A = 2(A_1 + A_3) =$ area of one hysteresis loop.

Thus the area of the loop is work done per unit volume in magnetising the material around the complete hysteresis loop. This energy lost in the form of heat and is known as hysteresis loss. Thus smaller the hysteresis loop area, smaller the hysteresis loss of magnetic material. According to C.P. Steinmetz, the area of the hysteresis loop is proportional to $B^n_{\max}$ *i.e.*

$$\frac{W}{V} \propto B^n_{\max}\ \text{J/m}^3 \quad \text{or} \quad \boxed{\frac{W}{V} = n_h\, B^n_{\max}\ \text{J/m}^3} \qquad \text{...(5.23)}$$

where n_h = hysteresis coefficient, and n = Steinmetz coefficient whose value ranges from 1.5 to 2.5.

The value of n_h and n depend upon the magnetic material under consideration. The value of n for pure iron, mild steel, cast iron, silicon steel etc. are taken as 1.6. In case, the frequency of magnetisation is f cycle per second, **the hysteresis loss per second per unit volume is given by**

$$\frac{W}{V} \times f = n_h\, f\, B^n_{\max} \quad \text{J/m}^3\ \text{sec.}$$

If total hysteresis loss is denoted by P, then

$$\boxed{P = n_h\, f\, B^n_{\max}\, V \quad \text{Joules/sec or Watts}} \qquad \text{...(5.24)}$$

where f = number of cycles per sec ; B_{max} = max flux density, in Wb/m^2

V = volume of the material, in m^3 ; n = 1.6 .... for most of the material.

It may be further mentioned that hysteresis loss is not the only loss in a magnetic material due to cyclic magnetisation but there is another loss called eddy current loss which often occurred in a transformer core. A little consideration will show that eddy current loss is given by

$$P_e = K_2 f^2 B_{max}^2 \quad ...(5.25)$$

and eddy current loss per unit surface area is given by

$$P_e = \frac{t^2 B_{max}^2 \omega^2}{24 \rho} \text{ Watts} \quad ...(5.26)$$

where t = thickness of the slab or lamination ; ρ = Resistivity

alternatively,

$$P_e = n_e t^2 f^2 B_{max}^2 V \text{ Watts} \quad ...(5.27)$$

where n_e = eddy loss coefficient of the material = $\pi^2/6\rho$.

2. Retarded potential

4.36 RETARDED POTENTIALS

The scalar electric potential V and vector magnetic potential A were respectively dealt in preceding and in this chapter on the basis of the charges being fixed in position for V and on the basis of constant charge velocities or constant current for $\vec{A}$. These potentials were expressed as

$$V = \frac{Q}{4\pi\varepsilon_0 R} \text{ Volts} \quad \text{for a concentrated charge,} \quad ...(2.13)$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_s \frac{\rho_s \, \vec{ds}}{r} \text{ Volts} \quad \text{for a surface charge,} \quad ...(2.55)$$

or

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho \, dv}{r} \text{ Volts} \quad \text{for a volume charge,} \quad ...(2.56)$$

and

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{q\vec{v}}{r} \quad \text{for a moving concentrated charge,} \quad ...(4.244)$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_c \frac{I\,\vec{dl}}{r} \text{ wb/m} \quad \text{for a contour line of constant current.} \quad ...(4.85)$$

or

$$\vec{A} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}\,dv}{r} \text{ wb/m} \quad \text{for a volume distribution of current density.} \quad ...(4.87)$$

These potentials are usable for the dynamic case, where both [illegible]

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{V/m}$$

for the establishment of the electric field intensity vector $\vec{E}$. The use of $-\nabla V$ [illegible] *of the electric field intensity caused by the* **position** *of the charge* [illegible] $-(\partial \vec{A}/\partial t)$ *indicates the component of the electric field intensity caused* [illegible] *the vector magnetic potential. However, in the forms of equation 3.22 and eqn.* [illegible] **of the inherent delay, which may be of vital importance in the dynamic case,** [illegible] **of the respective potential at a point P situated a distance r for the cause of the** [illegible]

If the intermediate space over which effect is to be propagated through [illegible] molecular substance that would support bound charge polarization or magnetization, [illegible] can be accounted for by the knowledge that all electric and magnetic effects are [illegible] of c w.r.t. the receiver P (in case it is moving). The velocity c is velocity of [illegible] (such as light) in free space and and is equal to 3×10^8 metres/seconds. [illegible] substances where polarization and magnetization effects are pronounced [illegible] general case study. As such, we shall consider only space where the velocity is [illegible] ε and μ are ε_0 and μ_0. The retarded scalar potentials then can be expressed [illegible] time $(t - r/c)$. For the volume-charge density the expression for retarded scalar [illegible]

$$\boxed{V_{(p,t)} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{[\rho]_{(t - \frac{r}{c})}\, dv}{r}} \quad \text{[illegible]}$$

where $V_{(p,t)}$ = Scalar electric potential at the point P evaluated at the time t.

$[\rho]_{(t-\frac{r}{c})}$ = Charge at the source S [Fig. 4.44 (b)] evaluated at an earlier value of time [illegible]

r/c = Retardation time is that time for the effect to be propagated the distance r at the velocity c.

If the point P and also the source S are moving, the velocity C is that w.r.t. [illegible] P and r is the measure of distance from where P is at time t to where S was at [illegible] $(t - r/c)$.

Similarly in terms of these sametime, velocity and distance designations, [illegible] magnetic potential can be expressed for the volume distribution of current density as

$$\boxed{\vec{A}_{p,t} = \frac{\mu_0}{4\pi} \int_V \frac{[\vec{J}]_{(t-\frac{r}{c})}}{r}\, dv} \quad \text{[illegible]}$$

The gradient of the scalar potential is required in eqn. $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ V/m [illegible]

Taking the negative partial space derivative of eqn. 4.12 w.r.t. the radial distance r gives

$$-\nabla V_{(p,t)} = -\int_v \vec{a}_r \frac{\partial}{\partial r} \left\{ \frac{[\rho]_{(t-\frac{r}{c})}}{r} \right\} dv \quad \text{[illegible]}$$

$$= -\int_V \vec{a}_r \left\{ \frac{[\rho]_{(t-\frac{r}{c})}}{r^2} + \frac{1}{r} \frac{\partial [\rho]_{(t-\frac{r}{c})}}{\partial r} \right\} dv \quad \text{[illegible]}$$

The negative time derivative of the vector magnetic potential is required in eqn. 4.12. The negative partial derivative of $\vec{A}$ w.r.t. time t is related to the current density at the retarded time location as

$$-\frac{\partial \vec{A}_{(p,t)}}{\partial t} = -\frac{\mu_0}{4\pi}\int_V \frac{\partial}{\partial t}\left\{\frac{[\vec{J}]_{(t-\frac{r}{c})}}{r}\right\} dv \qquad \text{...[4.148 (e)]}$$

Let τ be the retarded time, then $\tau = t - r/c$ ...[4.148 (f)]

If r is not a function of time *i.e.* if the two points P and S (at which the current density $\vec{J}$ is changing) are not moving relative to each other, then

$$\partial \tau = \partial t \qquad \text{...[4.148 (g)]}$$

and hence

$$-\frac{\partial \vec{A}_{(p,t)}}{\partial t} = -\frac{\mu_0}{4\pi}\int_V \frac{1}{r}\left[\frac{\partial \vec{J}}{\partial \tau}\right]_{\tau = t - r/c} dv \qquad \because \tau = t - \frac{r}{c} \qquad \text{...[4.86 (h)]}$$

If the location and time variations of all charges in the system and likewise all current densities both in **position** and in their **time variation** are known, then components of $\vec{E}$ from eqn. 4.12 are evaluated by eqn. 4.148 (d) and eqn. 4.148 (e) or eqn 4.148 (h). In electromagnetic field hardly the locations of all charges and time variations of current densities are known. Hence many field problems are solved not by the use of these equation directly. *Rather they are solved by resorting to Maxwell's field equations at one location in the field and realising that time at this location is related to time at another location in the field where boundary conditions on* $\vec{E}$ and $\vec{H}$ *may be known by the retarded time* $(t - r/c)$.

The time varying potentials normally called **retarded potentials**, find their greatest application in radiation problems in which the distribution of the source is known approximately. Retarded potentials discussion applies to the very important case of a region extending to infinite with a linear, isotropic and homogeneous medium. In the above discussion eqn. 4.148 (a, b), $(t - r/c)$ denotes that, *for an evaluation of V or* $\vec{A}$ *at time t, the value of charge density* ρ *at* $(t - r/c)$ *should be used.*

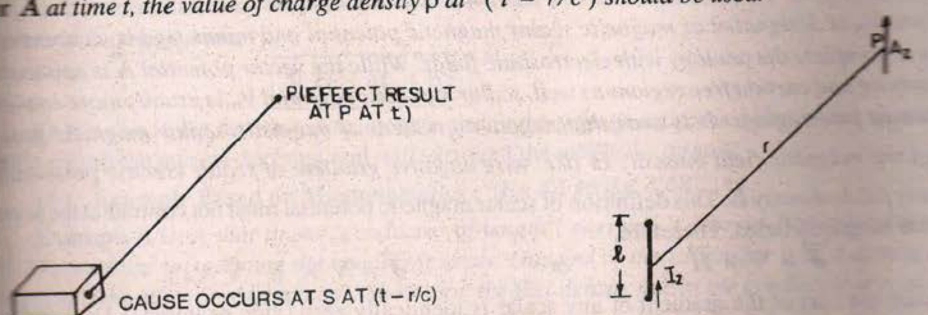


Fig. 4.44 (a) Retarded field at P caused by charge at S. Fig. 4.44. (b) Retarded potential from small current element.

One of the simplest examples illustrating the meaning of this retardation and, one that will be seen in the study of radiating system, is that of a very short wire carrying an a.c. current sinusoidally in time between two small spheres on which charges accumulate [Fig. 4.44 (b)]. For a filamentary current in a small wire, the difference in distance from point P to various points of a given cross-section of the wires are not important. Hence for any filamentary current.

$$\vec{A} = \frac{\mu}{4\pi}\int \frac{I\left(t - \frac{r}{c}\right) d\vec{l}}{r} \qquad \text{...[4.148 (i)]}$$

For a particular case of Fig. 4.85 current is in the z direction only and so also the $\vec{A}$. If l is small in comparison with r and wavelength λ, then on integration,

ELECTROMAGNETIC FIELDS AND WAVES

$$A_z = \frac{\mu}{4\pi} \cdot \frac{l I_z \left(t - \frac{r}{c}\right)}{r} = \frac{\mu l I_z \left(t - \frac{r}{c}\right)}{4 \pi r}$$

If the current in the small element has the form $I_z = I_m \sin \omega t$

Putting this in above eqn., we get,
$$\boxed{A_z = \frac{\mu l I_m \sin \omega \left(t - \frac{r}{c}\right)}{4 \pi r}} \quad ...[4.148]$$

The value of $\vec{A}$ and electric field may be derived. This will be carried out in the next chapter while dealing with the radiations.

Q.7 a. Explain the concept of "Displacement current". How is this current different from Conduction current? (8)

Answer:

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DUCTION CURRENT AND DISPLACEMENT CURRENT

Interpretation

der a parallel plate capacitor is connected to an alternating generator through e in series circuit as shown in fig. (16). We know that in a series circuit, the as the same value at all cross-sections. The n current which is due to electrical charges is same s-sections of the circuit except at all cross-sections B) of the dielectric inside capacitor. The dielectric h permittivity ε_0. It is important to mention here uction electrons (free electrons) are flowing into of the capacitor and forcing other free electrons e second plate. Therefore, *there is no flow of free through the dielectric*. In other words, we can say that there is a discontinuity rent in the space between the plates of a capacitor. Now, the question is that discontinuity of the current can be explained. In order to remove this uity J.C. Maxwell revised and extended the definition of current by ng the idea of *displacement current*.

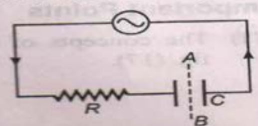

Fig. (16)

at any particular instant, q be the charge on capacitor plate. The conduction i_C) is defined as the time rate of flow of charge, *i.e.*,

$$i_C = \frac{dq}{dt} \quad \ldots(1)$$

have studied the electrical displacement D in dielectrics. The electrical ment D is given by

$$D = \sigma = \frac{q}{A} \quad \ldots(2)$$

is the surface charge density and A is the area of each plate.

m eq. (2), $$q = D\,A \quad \ldots(3)$$

stituting the value of q from eq. (3) in eq. (1), we get

$$i = \frac{d}{dt}(D\,A) = A\frac{dD}{dt} \quad \ldots(4)$$

xwell suggested that the term $A\left(\frac{dD}{dt}\right)$ should be considered as the current he dielectric. This current is called as *displacement current* and is denoted by i_d. ore,

$$\boxed{i_d = A\frac{dD}{dt}} \quad \ldots(5)$$

s important to mention here that current is not flowing when applied voltage is nt. It is flowing only when applied voltage is changing.

nsequently, the displacement current density $\mathbf{J}_d$ is given by

$$\boxed{\mathbf{J}_d = \frac{d\mathbf{D}}{dt}} \quad \ldots(6)$$

he vector $\mathbf{D}$ may vary with space, hence,

b. Derive the Maxwell's equation for static and time varying electric field. (8)

Answer:

MAXWELL'S EQUATIONS

We have studied that when the electric and magnetic fields are changing very idly in space with time, then the varying electric fields give magnetic field and -versa. Maxwell in 1862 formulated the basic laws of electromagnetic in the form four fundamental equations. These equations are known as Maxwell's tromagnetic equations. These equations are based upon the well known laws such Gauss's law of electrostatic, Gauss's law of magnetostatic, Faraday's law of tromagnetic induction and Amere's circuital law.

The integral forms of these equations are given below :

(1) Word statement : *The total flux coming out of a closed surface is equal to net chagrge enclosed.*

$$\oint \mathbf{E} \cdot d\mathbf{S} = \left(\frac{q}{\varepsilon_0}\right) \quad \text{or} \quad \oint_S \mathbf{D} \cdot d\mathbf{S} = q \qquad \ldots(1)$$

(2) Word statement : *The surface integral of magnetic flux density over a closed ace is zero.*

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \ldots(2)$$

(3) Word statement : *The net e.m.f. induced in a closed path is equal to the ace integral of negative time rate of change of flux density over the surface bounded losed path.*

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad \ldots(3)$$

(4) Word statement : *The total mmf around any close path must be equal to the ace integral of conduction and displacement densities over the surface bounded by closed path.*

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

or

$$\oint \mathbf{H} \cdot d\mathbf{l} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

or $$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t}\, d\mathbf{S} \qquad (\because I = \int_S \mathbf{J} \cdot d\mathbf{S})$$

or $$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

The differential forms (point forms) of these equations are given below :

$\text{div. } \mathbf{E} = \frac{\rho}{\varepsilon_0}$	or	$\vec{\nabla} \cdot \mathbf{D} = \rho$
$\text{div. } \mathbf{B} = 0$	or	$\vec{\nabla} \cdot \mathbf{B} = 0$
$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	or	$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\text{curl } \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$	or	$\vec{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

6.8 DEVIATION OF MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

1. Maxwell's first equation

$$\vec{\nabla} \cdot \mathbf{D} = \rho$$

According to Gauss's law $\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}$

If ρ be the charge density and dV, the small volume considered, then

$$q = \int_V \rho \, dV$$

$\therefore$ $$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV$$

or $$\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = \int_V \rho \, dV$$

or $$\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV \qquad (\because \varepsilon_0 \mathbf{E} = \mathbf{D})$$

According to divergence theorem,

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int_V (\vec{\nabla} \cdot \mathbf{D})\, dV$$

From eqs. (1) and (2), we get

$$\int_V (\vec{\nabla} \cdot \mathbf{D})\, dV = \int_V \rho \, dV$$

$\therefore$ $$\boxed{\vec{\nabla} \cdot \mathbf{D} = \rho}$$

From eq. (a), $\vec{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ or $\text{div. } \mathbf{E} = \frac{\rho}{\varepsilon_0}$ ...(3)

(4.2) Maxwell's second equation

$$\vec{\nabla} \cdot \mathbf{B} = 0$$

According to Gauss's law for magnetism

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

Transforming the surface integral into volume integral by Gauss's divergence theorem, we have

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_V (\vec{\nabla} \cdot \mathbf{B})\, dV$$

$$\therefore \quad \int_V (\vec{\nabla} \cdot \mathbf{B})\, dV = 0$$

As the volume is arbitrary, the integral must be zero. Hence,

$$\boxed{\vec{\nabla} \cdot \mathbf{B} = 0} \quad \text{...(b)}$$

$$\text{div. } B = 0$$

Maxwell's third equation

$$\vec{\nabla} \times \mathbf{E} = -\left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

According to Faraday's law

$$\oint \mathbf{E} \cdot dl = -\frac{d\Phi_B}{dt} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$$

or

$$\oint \mathbf{E} \cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} \quad \text{...(4)}$$

Applying Stoke's theorem

$$\oint \mathbf{E} \cdot dl = \int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} \quad \text{...(5)}$$

From eqs. (4) and (5), we get

$$\int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} \quad \text{...(6)}$$

Eq. (6) is true for all surfaces, therefore,

$$\boxed{\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \quad \text{...(c)}$$

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

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4. Maxwell' fourth equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

According to Ampere's circuital law, the line integral of magnetic f over a closed path is equal to the total current enclosed by the path

$$\therefore \quad \oint \mathbf{H} \cdot d\mathbf{l} = I = i_c + i_d$$

where I is the total current.

The total current is the sum of conduction current i_c and displaceme *i.e.*, $I = i_c + i_d$. In terms of conduction current density $\mathbf{J}_c$ and displace density $\mathbf{J}_d$, i_c and i_d can be expressed as

$$i_c = \int_S \mathbf{J}_c \cdot d\mathbf{S} \quad \text{and} \quad i_d = \int_S \mathbf{J}_d \cdot d\mathbf{S}$$

Substituting these values in eq. (7), we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_c \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t}\, d\mathbf{S}$$

or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t}\, d\mathbf{S}$$

($\because$ Normally, conduction current $\mathbf{J}_c$ is represented by $\mathbf{J}$ and $\mathbf{J}_d = \partial$ is electric flux density)

Eq. (9) can be expressed as

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$$

According to Stoke's law

$$\oint \mathbf{H} \cdot d\mathbf{l} = (\vec{\nabla} \times \mathbf{H}) \cdot d\mathbf{S}$$

From eqs. (10) and (11), we get

$$\int_S (\vec{\nabla} \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$$

Taking derivative of both sides, we get

$$(\vec{\nabla} \times \mathbf{H}) = \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$$

Q.8. a. Define the terms "Virtual height", "Critical frequency" and "Skip distance". (8)

Answer:

The virtual height of an ionospheric layer is best understood with the aid of Figure 8-14. This figure shows that as the wave is refracted, it is bent down gradually rather than sharply. However, below the ionized layer, the incident and refracted rays follow paths that are exactly the same as they would have been if *reflection* had taken place from a surface located at a greater height, called the *virtual height* of this layer. If the virtual height of a layer is known, it is then quite simple to calculate the angle of incidence required for the wave to return to ground at a selected spot.

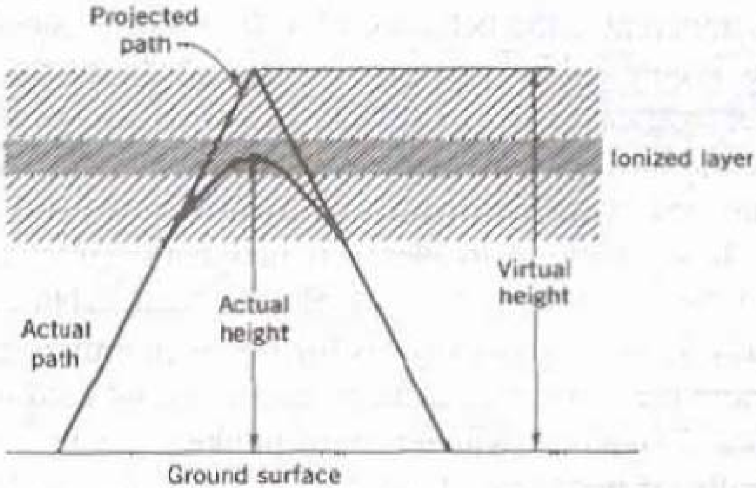


FIGURE 8-14 Actual and virtual heights of an ionized layer.

by a layer at vertical incidence. frequency which can be

11.7.3. Critical Frequency. The critical frequency of an ionized layer of the ionosphere is defined as *the highest frequency which can be reflected by a particular layer at vertical incidence. This highest frequency is called critical frequency for that particular layer and it is different for different layers. It is usually denoted by f_0 or f_c.* Critical frequency for the particular regular layer is proportional to the square root of the maximum electron density in the layer as shown below. From eqns. 11.40 and 11.41 we can write

$$\mu = \frac{\sin i}{\sin r} = \sqrt{1 - \frac{81 N}{f^2}} \quad \text{...(11.43)}$$

By definition, at vertical incidence

Angle of incidence $\angle i = 0$; $N = N_{\max}$ and $f = f_c$.

As the angle of incidence go on decreasing and reaches to zero, (*i.e.* vertical incidence) the electron density go on increasing and reaches to maximum electron density (N_m). Then the highest frequency that can be reflected back by the ionosphere is one for which refractive index μ becomes zero.

$$\therefore \quad \mu = \frac{\sin \theta}{\sin r} = \sqrt{1 - \frac{81 N_m}{f_c^{\,2}}} = 0 \quad \text{or} \quad 1 = \frac{81 N_m}{f_c^{\,2}} \quad \text{or} \quad f_c = \sqrt{81\,(N_m)} \quad \text{...[11.44 (a)]}$$

or $\boxed{f_c = 9\sqrt{N_m}}$

where f_c is expressed in MHz and N_m in per cubic metre. Thus if the maximum electron density N_m is known, the critical frequency can be calculated by eqn. 11.44. Of course critical frequency is the highest frequency which can be reflected by a particular layer *at vertical incidence but it is, not the highest frequency which will get reflected for any other angle of incidence.* The frequency that can be reflected from a layer is a function of angle of incidence (i) and is called maximum usable frequency MUF (to be seen next).

Thus critical frequency gives an idea that radio waves of frequency equal to or less than the critical frequency will certainly be reflected back by the ionospheric layer irrespective of the angle of incidence. Radio waves of frequency greater than critical frequency will also be returned to earth only

[illegible] SKIP DISTANCE

Radio wave radiated horizontally from a transmitter near the earth's surface is quickly absorbed [illegible] ground losses and hence only short distance communication is carried out by this horizontal [illegible] of ground or surface wave. Radio wave radiated at high angle may not be bent sufficiently at [illegible] layers to return to earth at all and hence escapes rather penetrates the layer. Thus radio [illegible] at shallow angle (*i.e.* angle between horizontal and high angle) just great enough to escape [illegible] the earth, will enter the lower layer, suffer attenuation, be bent at the upper layer and return [illegible]. In other words between, *the distance at which surface wave becomes negligible and the distance [illegible] the first wave returns to earth from the ionospheric layer, there is a zone which is not covered [illegible] (i.e. neither ground nor sky). This is called skip zone or area and the distance across it is the* [illegible]. Although, it is more usual to consider skip distance from the transmitter to the point [illegible] sky wave is received, as range of surface wave is always small.

Hence skip distance may be defined as

(i) The minimum distance from the transmitter at which a sky wave of given frequency is returned [illegible] earth by the ionosphere. It is represented by D as in the Fig. 11.26, or

(ii) The minimum distance from the transmitter to a point where sky wave of a given frequency is first received, or

(iii) The minimum distance within which a sky wave of given frequency fails to be reflected back, or

(iv) The minimum distance for which sky wave propagation just takes place and no sky wave propagation is possible for points nearer than this distance.

The higher the frequency, the higher the skip distance and for a frequency less than critical [illegible] of a layer skip distance is zero. As the frequency of a wave exceeds the critical frequency, the [illegible] the ionosphere depends upon the angle of incidence at the ionosphere as shown in Fig. 11.26 in [illegible] of different angle of incidence is shown.

As the angle of incidence at the ionosphere decreases, the distance from the transmitter, at which the ray returns to ground first decreases. This behaviour continues until eventually an angle of incidence is reached at which the distance becomes minimum. The minimum distance is called skip distance D (as with wave no. 2). With further decrease in angle of incidence, the wave penetrates the layer (as wave nos. 3 and 4) and does not return to earth. *Infact, skip distance is the distance skipped over by the sky wave.*

This happens because

(1) As the angle of incidence i is large (say for wave no.1), the eqn.

$$\mu = \sin i = \sqrt{1 - \frac{81\,N}{f^2}}$$

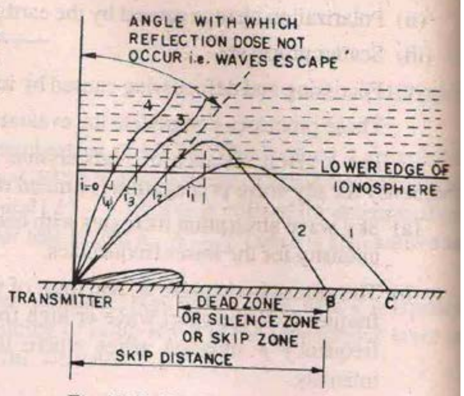


Fig. 11.26. Skip distance explanation

is satisfied with small electron density. This means μ is slightly less than unity and hence wave returns after slight penetration into the layer.

As the angle of incidence is further decreased (As in wave no. 2) sin i decrease still more and so also the μ, as N becomes comparatively more. Hence the wave penetrates still more before it reaches to earth.

Lastly when angle of incidence is small enough so that $\mu = \sin i$ can not be satisfied even by maximum electron density of the layer, then the wave penetrates (as the wave nos. 3 and 4).

The frequency which makes a given distance corresponds to the skip distance is the maximum usable frequency for those two points. If a receiver is placed with the skip distance no signals would be heard unless of course ground wave is strong enough as at A.

For a given frequency of propagation $f = f_{muf}$ the skip distance can be calculated from Eqn. 11.90 (b) in which D is the skip distance. Thus,

or $$\frac{f_{muf}}{f_c} = \sqrt{1 + \left(\frac{D}{2\,h}\right)^2} \quad \text{or} \quad \left(\frac{f_{muf}^2}{f_c}\right) - 1 = \left(\frac{D_{skip}}{2h}\right)^2$$

$$\boxed{D_{skip} = 2h\sqrt{\left(\frac{f_{muf}}{f_c}\right)^2 - 1}} \qquad \text{...(11.75)}$$

b. Assume the reflection takes place at a height of 400km and that the maximum density in the ionosphere corresponds to a 0.9 refractive index at 10 MHz. What will be the range for which the MUF is 10 MHz? Assume flat earth. (8)

Answer:

Example 11.12. *Assume that reflection takes place at a height of 400 km and that the maximum density in the ionosphere corresponds to a 0.9 refractive index at 10 MHz. What will be the range (assume flat earth) for which the MUF is 10 MHz.* (AMIETE, Principles of Comm. Engg., Dec. 1983)

Or

In the ionospheric propagation, consider that the reflection takes place at a height of 400 km and that the maximum density in the ionosphere corresponds to a refractive index of 0.9 at a frequency of 10 MHz. Determine the ground range for which this frequency is the MUF. Take the earth's curvature into consideration. (AMIETE, Principle of Communication Engineering, June 1984)

Solution. We know that

$$\mu = \sqrt{1 - \frac{81\,N}{f^2}} \quad \text{and} \quad D_{skip} = 2\,h\sqrt{\left(\frac{f_{muf}}{f_c}\right)^2 - 1}$$

Given, $h = 400$ km. ; $\mu = 0.9$; $f_{muf} = 10$ MHz ; $f = 10$ MHz and $D_{skip} = D_{range} = ?$

Putting these values in above eqn. we get,

$$0.9 = \sqrt{1 - \frac{81\,N_{max}}{f^2}} \quad \text{or} \quad 0.81 = 1 - \frac{81\,N_{max}}{f^2} \quad \text{or} \quad \frac{81\,N_{max}}{f^2} = 1 - 0.81 = 0.19$$

or $$N_{max} = \frac{0.19 \times f^2}{81} = \frac{0.19 \times (10 \times 10^6)^2}{81} = \frac{0.19 \times 10^{14}}{81} = 0.0023456 \times 10^{14} = \mathbf{23.456 \times 10^{10}\ m^{-3}}$$

Hence, $$f_c = 9\sqrt{N_{max}} = 9\sqrt{23.456 \times 10^{10}} = 9 \times 4.8431 \times 10^5\ \text{Hz} = 43.588 \times 10^5\ \text{Hz} = \mathbf{4.3588 \times 10^6\ Hz}$$

Case I. When earth is flat.

$$D_{skip} = 2 \times 400\sqrt{\left(\frac{f_{muf}}{f_c}\right)^2 - 1} = 800\sqrt{\left(\frac{10 \times 10^6}{4.3588 \times 10^6}\right)^2 - 1} = 800\sqrt{\left(\frac{10}{4.3588}\right)^2 - 1}$$

$$= 800\sqrt{(2.2942)^2 - 1} = 800\sqrt{5.2633 - 1} = 800\sqrt{4.2633} = 800 \times 2.0647 = \mathbf{1651.76\ km\ Ans.}$$

Case II. When the earth's curvature is taken into account, then R = 6370 km = Radius of the earth, h = height of reflecting layer from the earth.

In this case vide eqn. 11.99

$$\frac{f_{muf}}{f_c} = \sqrt{\frac{D^2}{4\left(h + \frac{D^2}{8R}\right)^2} + 1} \quad \text{or} \quad \frac{D^2}{4\left(h + \frac{D^2}{8R}\right)^2} = \left(\frac{f_{muf}}{f_c}\right)^2 - 1$$

$$D = 2\left(h + \frac{D^2}{8R}\right)\sqrt{\left(\frac{f_{muf}}{f_c}\right)^2 - 1}$$

Now putting the values, we have

$$= 2\left(400 + \frac{1651.76 \times 1651.76}{8 \times 6370}\right)\sqrt{4.2633} = \left(800 + \frac{272831.1}{25480}\right) \times 2.0647 \quad | \text{ From Case I}$$

$$= (800 + 10.707656) \times 2.0647 = 80.707656 \times 2.0647 = 1673.868\ \text{km} \cong \mathbf{1673.86\ km\ Ans.}$$

Q.9 a. State Babinet principal and explain how it gives rise to the concept of complementary antenna? (8)

Answer:

BABINET'S PRINCIPLE AND COMPLEMENTARY ANTENNAS

One may enquire whether there is any relation between wire antenna and aperture antenna, the answer can be answered better by first introducing Babinet's principle of optics. The Babinet's (Ba-bi-nay's) in optics states that *"When the field behind a screen with an opening is added to the field of a complementary structure, the sum is equal to the field when there is no screen"*.

Babinet principle in optics does not consider polarization, which is so vital in antenna theory. It deals primarily with absorbing screens. An extension of Babinet's principle, which induces polarization and the more practical conducting screens, was introduced by Booker. By introduction of Babinet's principle many of the problems of slot antennas can be reduced to situation involving complementary wire antennas for which solutions have already been obtained.

The Babinet's principle may be illustrated by considering the following example with three cases.

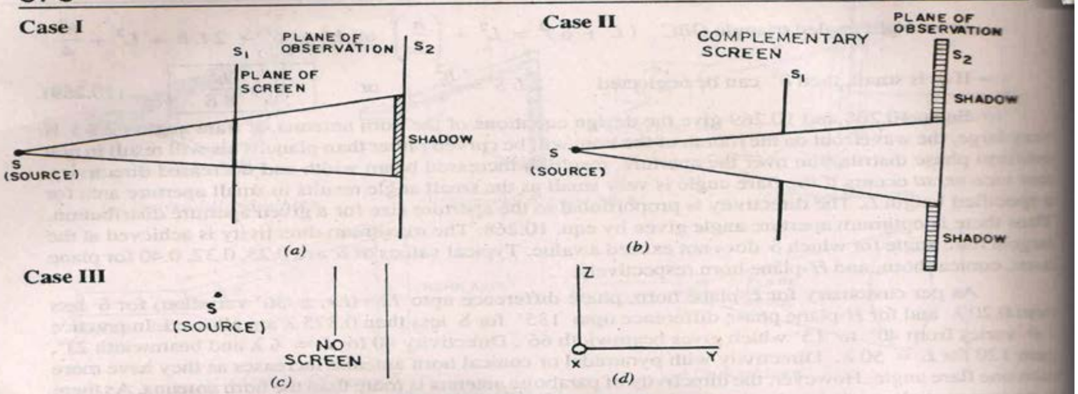


Fig. 10.73. Babinet's principle.

Let a source and two imaginary planes are arranged as shown in Fig. 10.73 in which the first plane is a plane of screens S_1 and the plane is a plane of observation S_2. Now three cases arise.

Case I. Let a perfectly absorbing screen be placed in plane S_1, then in plane S_2 ,, there is a region of shadow as shown. Let the field behind this screen be same function of f_1 $(x\,y\,z)$ *i.e.*

$$\boxed{F_1 = f_1\ (x\,y\,z)} \quad ...(10.272)$$

Case II. Let the first screen S_1 be replaced by its complementary screen and the field behind it be given by

$$\boxed{F_2 = f_2\,(x\,y\,z)} \quad ...(10.273)$$

Case III. Let there is no screen present, then the field is given by

$$\boxed{F_3 = f_3\ (x\,y\,z)} \quad ...(10.274)$$

Babinet's principle then states that at the same point $(x\,y\,z)$

$$F_3\ (x\,y\,z) = F_1\ (x\,y\,z) + F_2\ (x\,y\,z)$$

or

$$\boxed{F_3 = F_1 + F_2} \quad ...(10.275)$$

The source may be a point as in the above example or a distribution of sources. The principle applies not only to points in the plane of observation S_2 as outlined in Fig. 10.73 but also to any point behind screen S_1. The principle is obvious enough for shadow (case I), it is also true when diffraction is taken into accounts.

The correctness of this valid statement (eqn. 10.275) can be verified easily for the simple cases of complementary screens consisting of semi-infinite absorbing planes.

In electromagnetics at radio frequencies, thin perfectly absorbing screens are not available, even approximately and one is concerned with *conducting screens* and *vector fields* for which polarization plays an important role. As such the simple statement of optics could not be expected to apply but an extension of the principle, valid for conducting screens and polarized fields has been formulated by H.G. Booker.

As an illustration of Booker's extension of Babinet's principle, let us consider the following three cases shown in Fig. 10.74. The source (s) in all the three cases is a short dipole, theoretically infinitesimal dipole.

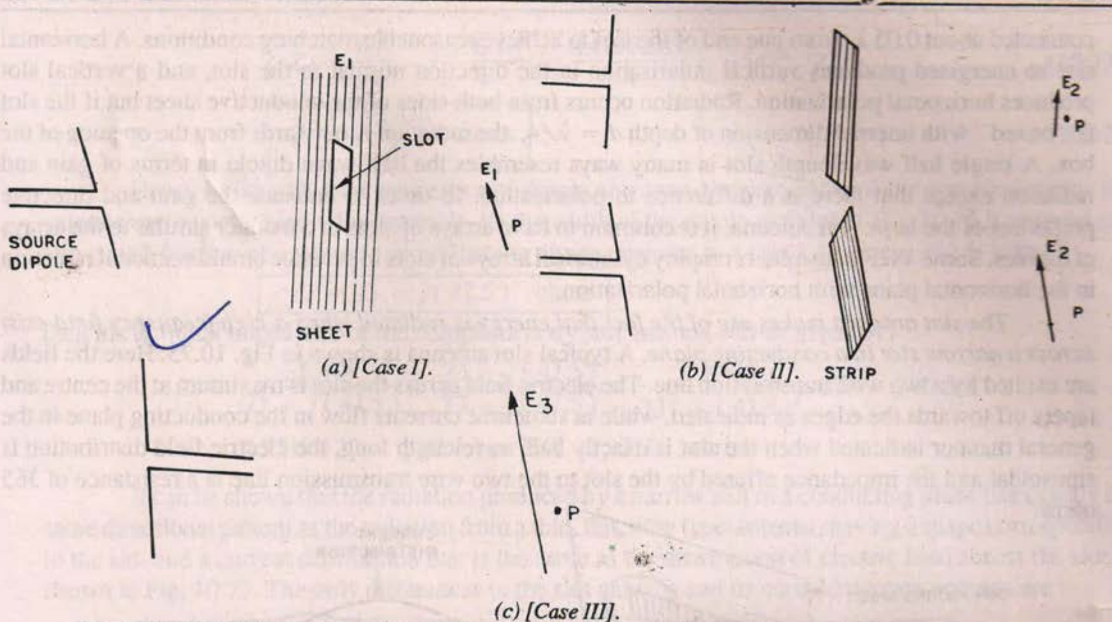


Fig. 10.74. Extension of Babinet principle for slot of infinite metal sheet and the complementary metal strip.

Case I. The dipole is horizontal and original screen is an infinite, perfectly conducting.plane, infinitesimally thin sheet with a vertical slot cut out. At a point P behind the screen the field is E_1.

Case II. In this case the original screen is replaced by the complementary screen consisting of a perfectly conducting, plane infinitesimally thin strip of the same dimensions as the slot in the original screen. Besides, the dipole is source and is turned vertical so that $\vec{E}$ and $\vec{H}$ are interchanged. At the same point P, behind the screen the field is E_2 .

Alternatively, the dipole source is turned horizontal and so also the strip.

Case III. In this case, no screen is placed and the field at point P is $\vec{E}_3$.

According to Babinet's principle $E_1 + E_2 = E_3$ or $\boxed{\dfrac{E_1}{E_3} + \dfrac{E_2}{E_3} = 1}$ ...(10.276)

The principle may also be applied to points in front of the screens. In case I, a large amount of energy may be transmitted through the slot so that $E_1 \simeq E_3$. In such situation the complementary dipole (case II) acts like a reflector and E_2 is very small.

Using Booker's extension, it can be shown that if a screen and its complement are immersed in a medium with an intrinsic impedance η and have terminal impedances of Z_S (screen) and Z_C (complementary) respectively, then the impedances are related by

$$\boxed{Z_S\, Z_C = \frac{\eta^2}{4}} \qquad \text{...(10.277)}$$

In order to obtain the impedance Z_C of the complementary dipole in practical arrangement a gap must be introduced to represent the feed points.

b. Write short note on: **(8)**

(i) Marconi antenna & Hertz antenna (ii) YAGI_UDA antenna.

Answer:

10.42. HERTZ AND MARCONI ANTENNAS

These are the two fundamental types of simple antennas and all other types of simple antennas may be considered as derivatives of one or the other of these.

The λ/2 or Hertz Antenna: is perhaps the most popular antenna in high frequency antennas. *An antenna complete in itself and capable of self oscillation, such as half or full wavelength (λ/2 or λ) known as a Hertz antenna.*

The λ/2 or Marconi Antenna. When an antenna utilizes the ground (earth) as part of its resonant circuit, it is a Marconi antenna. A quarter wave antenna (λ/4) is an example of Marconi antenna, where the ground operates as the missing quarter wavelength. Most of the low and medium frequency antennas are of Marconi types. The invention of the λ/4 earthed antenna in which the earth is one plate of the condenser, is considered to be the most important contributions of Marconi to the radio engineering. Marconi produced lofty and efficient antenna system from the short Hertzian radiator and achieved a long distance communication with low radio frequency.

10.50. YAGI-UDA ANTENNA

Yagi-uda or simply Yagi (as generally but less correctly called) antennas or Yagis are the most high gain antennas and are known after the names of Professor S. Uda and H. Yagi. The antenna was invented and described in Japanese by the former some time around 1928 and afterwards it was described by H. Yagi in English. Since the Yagi's description was in English so it was widely read and thus it became customary to refer this array as Yagi antenna, although he gave full credit to professor Uda. Accordingly a more appropriate name the *Yagi-Uda* antenna is adopted following the practice.

It consists of a driven element, a reflector and one or more directors *i.e.* Yagi-Uda antenna is an array of a *driven element* (or active element where the power from the T_X is fed or which feeds received power to the R_X) and one or more *parasitic elements* (*i.e.* passive elements which are not connected directly to the transmission line but electrically coupled). The driven elements is a resonant half-wave dipole usually of metallic rod at the frequency of operation. The parasitic elements of continuous metallic rods are arranged parallel to the driven element and at the same line of sight level. They are arranged collinearly and close together as shown in Fig. 10.63 with one reflector and one director. The optical equivalent is also shown.

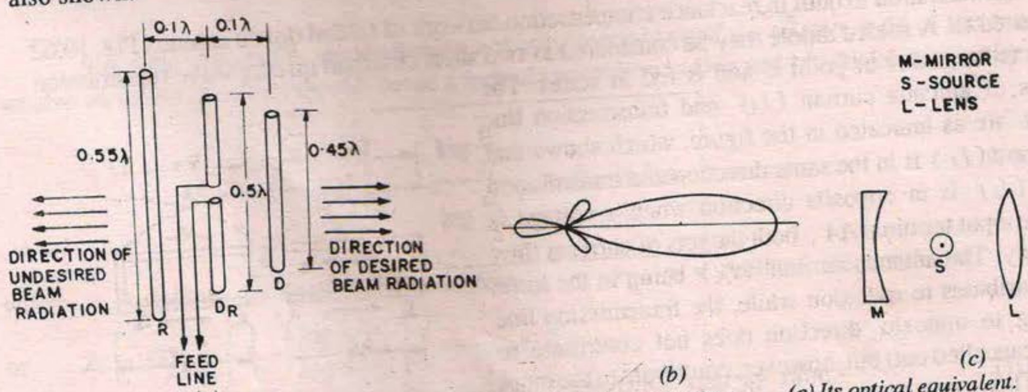


Fig. 10.63 (a) Yagi-Uda antenna, (b) Its radiation pattern, (c) Its optical equivalent.
R ≡ Reflector (Parasitic element) ; D_R = Driven element ; D = Driector (Parasitic element).

The parasitic elements receive their excitation from the voltages induced in them by the current flow in the driven element. The phase and currents flowing due to the induced voltage depend on the spacing between the elements and upon the reactance of the elements (*i.e.*, length). The reactance may be varied by dimensioning the length of the parasitic element. The spacing between driven and parasitic elements that are usually used, in practice, are of the order of $\lambda/10$ *i.e.* 0.10λ to 0.15λ. The parasitic element in front of driven, element is known as *director* and its number may be more than one, whereas the element in back of it is known as *reflector*. Generally both directors and reflectors are used in the same antenna. The reflector is 5% more and director is 5% less than the driven element which is $\lambda/2$ at resonant frequency. In practice, for 3-element array of Yagi antenna the following formulae gives lengths which work satisfactorily.

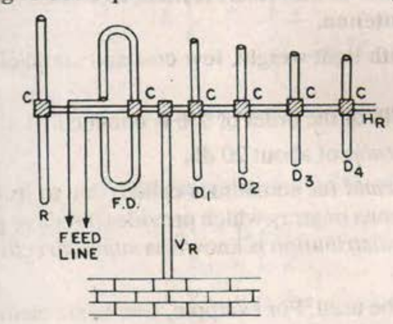
$$\text{Reflector length} = \frac{500}{f\,(\text{MHz})}\ \text{feet} \quad \text{...[10.246 (a)]}$$

$$\text{Driven element length} = \frac{475}{f\,(\text{MHz})}\ \text{feet} \quad \text{...[10.246 (b)]}$$

$$\text{Director length} = \frac{455}{f\,(\text{MHz})}\ \text{feet} \quad \text{...[10.246 (c)]}$$

Eqn. 10.246 provides average length of Yagi antenna determined experimentally for elements of length/diameter ratio of 200 to 400 and spacing from 0.10λ to 0.20λ. The parasitic elements can be clamped on a metallic support rod because at the middle of each parasitic element, the voltage is minimum *i.e.* there exists a voltage node. Even driven element may also be clamped if it is shunt feed. The clamping over the support rod makes a rigid mechanical structure.

Further use of parasitic elements in conjunction with driven element causes the dipole impedance to fall well below 73 Ω. It may be as low as 25 Ω and hence it becomes necessary to use either shunt feed or folded dipole so that input impedance could be raised to a suitable value, to match the feed cable. While using folded dipole the continuous rod may also be clamped to the support as shown in Fig. 10.64.



R = Reflector
FD = Folded dipole
$D_1\,D_2\,D_3\,D_4$ = Directors
VR = Vertical rod to support horizontal rods
HR = Horizontal rod to support elements
C = clamps

Fig. 10.64. 6 Elements Yagi antenna with folded dipole.

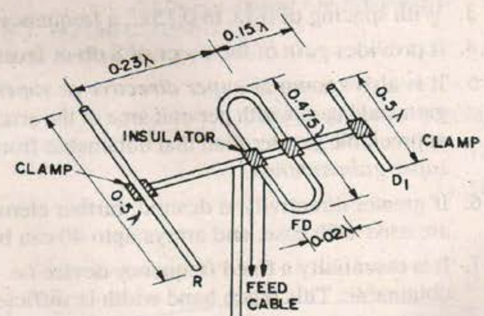


Fig. 10.65. A typical Television Yagi Antenna.

A typical 3 elements yagi antenna suitable for TV reception of moderate field strength is shown in Fig. 10.65. Further addition of directors can be done at intervals of 0.15λ *i.e.* to increase the gain even upto 12 db as is required in for fringe area reception. For example, 11 elements Yagi antenna the lengths of $D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9$ are respectively 0.427λ, 0.40λ, 0.38λ, 0.36λ, 0.32λ, 0.304λ, and 0.29λ.

TEXT BOOK

I. Engineering Electromagnetics, W. H. Hayt and J. A. Buck, Seventh Edition, Tata McGraw Hill, Special Indian Edition 2006

II. Electronic Communication Systems, George Kennedy and Bernard Davis, Fourth Edition (1999), Tata McGraw Hill Publishing Company Ltd