Q.2 a. Use Cauchy_Riemann equations to find v where $u = 3x^2y - y^3$. **(8)**

Answer:

Q.2 (a) Given $u = 3x^2y - y^3$ --(i)

Differentially (i) partially w.r.t. x & y respectively

$$\frac{\partial u}{\partial x} = 6xy = \phi_1(x,y) \quad \text{--(ii)}$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 = \phi_2(x,y) \quad \text{--(iii)} \quad \boxed{2 \text{ Marks}}$$

Put $x = z$, $y = 0$ in (ii) & (iii), we get

$$\phi_1(z,0) = 0, \quad \phi_2(z,0) = 3z^2$$

Hence by Milne-Thompson method,

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + ci \quad \boxed{4 \text{ marks}}$$

$$u + iv = -iz^3 + ic = -i(x+iy)^3 + ic$$

$$= -i(x^3 + 3ix^2y - 3xy^2 - iy^3) + ic \quad \boxed{6 \text{ marks}}$$

equating imaginary parts both sides -

$$v = 3xy^2 - x^3 + c \quad \boxed{\text{Ans}} \quad \boxed{8 \text{ marks}}$$

b. Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2 + y^2 - 4x = 0$ onto the straight line $4u + 3 = 0$, and explain why the curve obtained is not a circle. **(8)**

Answer:

(b) The Given trasformation is

$$w = \frac{2z+3}{z-4} \quad \text{---(i)}$$

Inverse transformation of (i) is

$$z = \frac{4w+3}{w-2} \quad \text{---(ii)}$$

Thus

$$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2} \quad \text{---(iii)} \quad \boxed{2 \text{ Marks}}$$

Given equation of circle $x^2+y^2-4x=0$ may be written as

$$z\bar{z} - 2(z+\bar{z}) = 0 \quad \text{--(iv)} \quad \boxed{4 \text{ Marks}}$$

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Put the values of z & $\bar{z}$ from (ii) & (iii)

$$\frac{4w+3}{w-2}\cdot\frac{4\bar{w}+3}{\bar{w}-2}-2\left(\frac{4w+3}{w-2},\frac{4\bar{w}+3}{\bar{w}-2}\right)=0$$

or $16w\bar{w}+12w+12\bar{w}-9-2(4w\bar{w}-8w+3\bar{w}-6$
$+4w\bar{w}-8\bar{w}+3w-6)=0$ $\boxed{6\text{ marks}}$

or $22(w+\bar{w})+33=0$

$44u+33=0$

or $4u+3$ --- (v) $\boxed{7\text{ marks}}$

which is the eq of straight line

Explanation: The given transformation transform the given circle into a straight line, this is possible under a bilinear transformation since we regard a straight line as a particular case of a circle. $\boxed{8\text{ marks}}$

Q.3 a. Prove that the function $f(z)=u+iv$ **where** $f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$ $(z\neq 0), f(0)=0$ **is continuous and that Cauchy_Riemann equations are satisfied at the origin, yet** $f'(z)$ **does not exist there. (8)**

Answer:

Q. 3 (a) Given $u + iv = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$u = \dfrac{x^3 - y^3}{x^2+y^2}$ and $v = \dfrac{x^3+y^3}{x^2+y^2}$ [1 marks]

u & v are rational and finite for all values of $z \neq 0$, so u & v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ is continuous where $z \neq 0$. [2 Marks]

At the origin $u = 0$, $v = 0$

Therefore u & v both ...nuous at the origin.

Consequently $f(z)$ is continuous ... the origin.

We have $u_x = \lim_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = 1$, similarly

$u_y = -1$, $v_x = 1$, $v_y = 1$ [4 marks]

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

Hence C-R equations satisfied at $z = 0$

Again $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \left[\dfrac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)} \right]$

Let $z \to 0$ along $y = x$

$f'(0) = \lim_{x \to 0} \dfrac{(x^3 - x^3) + i(x^3+x^3)}{(x^2+x^2)(x+ix)} = \dfrac{1}{2}(1-i)$ [6 marks]

Further let $z \to 0$ along $y = 0$ then $f'(0) = 1 + i$

Thus $f'(0)$ is not unique

Hence $f'(z)$ does not exist at the origin [8 Marks]

b. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ **in a Laurent's series valid for the regions**
(i) $|z| < 1$ **(ii)** $1 < |z| < 3$. **(8)**

Answer:

(3)

(b) $$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

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(i) For $|z| < 1$, we have $\frac{|z|}{3} < \frac{1}{3} < 1$

$$\therefore f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$ $\boxed{2 \text{ marks}}$

$$= \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{21}z^3 + \cdots$$ $\boxed{4 \text{ marks}}$

(ii) for $|z| > 1$, we have $\frac{1}{|z|} < 1$

and $|z| < 3$, we have $\frac{|z|}{3} < 1$

Hence

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$ $\boxed{6 \text{ marks}}$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)$$

$$= \cancel{\frac{1}{6}} \cancel{- \frac{z}{18}} \cancel{+ \frac{z^2}{54}} \cancel{- \frac{z^3}{}}$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} + \cdots - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \cdots$$ $\boxed{8 \text{ marks}}$

Q.4 a. If $(\varepsilon) = \int_C \frac{6z^2+5z+2}{z-\varepsilon}\, dz$ **, where** C **is the ellipse** $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ **, find the values of** $F(i), F'(-i)$ **and** $F''(-1)$**. (8)**

Answer:

Q. 4 (a) Take $f(z) = 6z^2 + 5z + 2$ which is analytic within and on the ellipse C. Also $\xi = i, -i, -1$ all lie inside C.

By Cauchy's integral formula, we have

$$F(\xi) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-\xi}$$ [2 marks]

$$\Rightarrow \int_C \frac{f(z)\,dz}{z-\xi} = 2\pi i\,F(\xi)$$

$$\Rightarrow \int_C \frac{6z^2+5z+2}{z-\xi} = 2\pi i\,(6\xi^2+5\xi+2)$$ [4 marks]

$$\Rightarrow F(\xi) = 2\pi i(6\xi^2+5\xi+2) \quad \text{---(i)}$$

$$\therefore F'(\xi) = 2\pi i\,(12\xi+5) \quad \text{---(ii)}$$

and $$F''(\xi) = 2\pi i\,(12) \quad \text{---(iii)}$$

Put $\xi = i$ in (i), $\xi = -i$ in (ii) & $\xi = -1$ in (iii), [6 Marks]

$$F(i) = 2\pi(-5-4i)$$

$$F'(-i) = 2\pi(12+5i)$$

$$F''(-1) = 24\pi i$$ [Ans] [8 Marks]

b. If F $= xy^2 i + 2x^2yzj - 3yz^2k$ **, find (i) Div F (ii) Curl F at the point (1, -1, 1) . (8)**

Answer:

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(b) (i) $\text{Div } F = \nabla \cdot F$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xy^2 i + 2x^2yz\,k - 3yz^2 k\right)$$

$$= \cancel{yz}\ \ y^2 + 2x^2z - 6yz$$ [2 marks]

$\therefore$ At the point $(1, -1, 1)$,

$\text{Div } F = 9$ (Ans) [4 marks]

(ii) $\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$ [6 marks]

$$= -(3z^2 + 2x^2y)\,i + (4xyz - 2xy)\,k$$

$\therefore$ At the point $(1, -1, 1)$

$\text{Curl } F = -i - 2k$ (Ans) [8 marks]

Q.5 a. Find the directional derivative of $\emptyset = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$ **at the point P(3,1,2) in the direction of the vector** $yzi + xzj + xyk$. **(8)**

Answer:

Q. 5 (a) Here $\phi = (x^2+y^2+z^2)^{-1/2}$

$\therefore \frac{\partial \phi}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial \phi}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$, $\frac{\partial \phi}{\partial z} = -z(x^2+y^2+z^2)^{-3/2}$ $\boxed{2 \text{ marks}}$

$\therefore \nabla\phi = \text{Grad } \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$= -(x^2+y^2+z^2)^{-3/2}(ix+jy+kz)$

$\nabla\phi = -\frac{(3i+j+2k)}{14\sqrt{14}}$ at point $(3,1,2)$ $\boxed{4 \text{ marks}}$

Now $\hat{a}$ = unit vector in the direction of vector $yzi+xzj+xyk$

$= \frac{yzi+xzj+xyk}{\sqrt{y^2z^2+x^2z^2+x^2y^2}} = \frac{2i+6j+3k}{7}$ at $(3,1,2)$ $\boxed{6 \text{ marks}}$

$\therefore$ Required directional derivative in the given direction $= \hat{a}\cdot\text{grad }\phi$

$= -\left(\frac{2i+6j+3k}{7}\right)\cdot\left(\frac{3i+j+2k}{14\sqrt{14}}\right)$

$= -\frac{18}{98\sqrt{14}}$

$= -\frac{9}{49\sqrt{14}}$ $\boxed{\text{Ans}}$ $\boxed{8 \text{ marks}}$

b. Evaluate $\iint_S F.n\, dS$ with the help of Gauss's divergence theorem for $F = 4xi - 2y^2j + z^2k$ taken over the region S bounded by $x^2+y^2=4, z=0$ and z = 3. **(8)**

Answer:

(b) Div $F = \nabla\cdot F = 4 - 4y + 2z$

By Gauss Divergence theorem, we have

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv \qquad \boxed{1 \text{ Marks}}$$

$$\iint_S F\cdot n\,ds = \iiint_V 2(2-2y+z)\,dx\,dy\,dz \qquad \boxed{2 \text{ Marks}}$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 2(2-2y+z)\,dz\,dx\,dy \qquad \boxed{3 \text{ Marks}}$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(6-6y+\tfrac{9}{2}\right)dx\,dy$$

$$= 2\int_{-2}^{2}\frac{3}{2}\left(7y-2y^2\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx \qquad \boxed{5 \text{ Marks}}$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx = 84\int_0^2\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{2}\cdot 4\cdot\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 \qquad \boxed{7 \text{ marks}}$$

$$\iint_S F\cdot n\,ds = 84\pi \quad \boxed{\text{Ans}} \qquad \boxed{8 \text{ Marks}}$$

Q.6 a Solve the partial differential equation : (8)

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$$

Answer:

Q. 6 (a) Lagrange's auxiliary equations are :

$$\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} \quad \text{--(i)} \qquad \boxed{1 \text{ Mark}}$$

Using x, y, z as multipliers, then eqn each fraction of (i) $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$ $\boxed{3 \text{ Marks}}$

$$\therefore\; x^2+y^2+z^2 = c_1 \quad \text{--(ii)}$$

taking last two members of (1), we get

$$y\,dy - (z\,dy + y\,dz) - z\,dz = 0$$

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{c_2}{2}$$

$$\Rightarrow\; y^2 - 2yz - z^2 = c_2 \quad \text{--(iii)} \qquad \boxed{6 \text{ Marks}}$$

Hence required solution is

$$\phi(x^2+y^2+z^2,\; y^2-2yz-z^2) = 0 \qquad \boxed{8 \text{ Marks}}$$

b. Solve by Charpit's method : (8)

$z = px + qy + pq$

Answer:

(b) Given $z = px + qy + pq$

$f \equiv px + qy + pq - z = 0$

$\therefore \frac{\partial f}{\partial x} = p;\ \frac{\partial f}{\partial y} = q;\ \frac{\partial f}{\partial z} = -1;\ \frac{\partial f}{\partial p} = x + q;\ \frac{\partial f}{\partial q} = y + p$ [2 marks]

Put the values in the charpit's auxiliary eqn's

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{-p(x+q) - q(y+p)} = \frac{dx}{-p} = \frac{dy}{-q}$$ [5 marks]

$\therefore\ p = a,\ q = b$

$z = ax + by + ab$

where a & b are arbitrary constants. [8 marks]

Q.7 a. Using Newton's divided difference formula, find $f(4)$ if : (8)

x	:	–4	–1	0	2	5
$f(x)$	:	1245	33	5	9	1335

Answer:

Q.7 (a) Divided Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–4	1245				
		–404			
–1	33		94		
		–28		–14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

[3 marks]

$f(4) = 1245 + (4+4)\times -404 + (4+4)(4+1)\times 94 +$
$(4+4)(4+1)4\times(-14) + (4+4)(4+1)\times 4\,(4-2)\times 3$

$= 1245 - 3232 + 3760 - 2240 + 960$ [6 marks]

$f(4) = 5965 - 5472 = 493$ [8 marks]

b. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3rd and 3/8th rule. Hence obtain the approximate value of π in each case. **(8)**

Answer:

(b) Take $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$5/6$	1
y	1	0.97297	0.9	0.8	0.69231	0.59016	0.5

[2 marks]

By Simpson's $\frac{1}{3}$rd Rule

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots y_{n-1}) + 2(y_2+y_4+\cdots)\right]$$

[4 marks]

$$= 0.785397$$

By Simpson's $3/8$th rule

$$I = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

[6 marks]

$$= 0.785395$$

value of integral $I = \pi/4$

By Simpson's $\frac{1}{3}$ Rule $\pi = 3.141588$ (Approx)

By —— $3/8$th Rule $\pi = 3.141580$ (Approx.) [8 Marks]

Q.8 a. In an examination the number of candidates who secured marks between certain limits were as follows :

Marks	0-19	20-39	40-59	60-79	80-99
No. of Candidates	41	62	65	50	17

Estimate the number of candidates getting marks less than 70. **(8)**

Answer:

Q.8 (a) Difference table

Marks Less than x	Number of candidates $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
19	41	62			
39	103	65	3	−18	
59	168	50	−15	−18	0
79	218	17	−33		
99	235				

[3 Marks]

Here $h = \frac{x-a}{h} = \frac{70-19}{20} = 2.55$

By Newton's forward interpolation formula,

$f(70) = 41 + 2.55 \times 62 + \frac{2.55 \times 1.55}{2!} \times 3 + \frac{2.55 \times 1.55 \times 0.55}{3!} \times -18$

$= 41 + 158.1 + 5.92 - 6.52 = 198.5$ [7 Marks]

$f(70) = 198$ (nearly) Ans. [8 Marks]

b. From a pack of 52 cards, 6 cards are drawn at random. Find the probability of the following events : (8)
(i) Three are red and 3 are black cards
(ii) Three are kings and 3 are queens

Answer:

(b) The total number of ways in which 6 cards can be drawn $= {}^{52}C_6$ [1 Mark]

$\therefore$ number of elements in the sample space S are $n(S) = {}^{52}C_6$

(i) $n(E_1)$ = Number of ways in which 3 red and 3 black cards can be selected [3 Marks]

$= {}^{26}C_3 \times {}^{26}C_3$

$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6}$ [5 Marks]

(ii) $n(E_2)$ = event that 3 kings and 3 queens can be selected

$= {}^4C_3 \times {}^4C_3$

$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^4C_3 \times {}^4C_3}{{}^{52}C_6}$ [8 Marks]

Q.9 a. A continuous random variable X has the density function
$(x) = 3x^2, 0 \le x < 1$.
Find *a* and *b*, when
(i) $P(X \le a) = P(X > a)$; **(ii)** $P(X > b) = 0.05$ **(8)**

Answer:

Q.9 (a) Since total probability is always 1 (one), so

(i) $P(X \le a) = \frac{1}{2} = P(X > a)$

Now $P(X \le a) = \frac{1}{2} \Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$ $\boxed{\text{2 marks}}$

$\Rightarrow \int_0^a 3x^2\,dx = \frac{1}{2}$

$\Rightarrow a^3 = \frac{1}{2}$ or $a = \left(\frac{1}{2}\right)^{1/3}$ $\boxed{\text{4 Marks}}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^1 f(x)\,dx = \frac{1}{20}$

$\Rightarrow \int_b^1 3x^2\,dx = \frac{1}{20}$ $\boxed{\text{6 marks}}$

$\Rightarrow 1 - b^3 = \frac{1}{20}$; $b^3 = \frac{19}{20}$

$\Rightarrow b = \left(19/20\right)^{1/3}$ $\boxed{\text{8 marks}}$

b. Fit a Poisson's distribution to the following and calculate theoretical frequencies ($e^{-0.5} = 0.61$) :

Deaths	**:**	**0**	**1**	**2**	**3**	**4**
Frequency	**:**	**122**	**60**	**15**	**2**	**1**

(8)

Answer:

(b) Here total frequency

$$N = \Sigma f = 122 + 60 + 15 + 2 + 1 = 200$$

$$\text{mean } (m) = \frac{\Sigma f x}{N} = \frac{122\times 0 + 60\times 1 + 15\times 2 + 2\times 3 + 1\times 4}{200}$$

$$\boxed{m = 0.5}$$ $\boxed{2 \text{ marks}}$

Now $e^{-m} = e^{-0.5} = 1 - 0.5 + \frac{(0.5)^2}{2} - \dots = 0.61$

Therefore required Poisson distribution

$$= N e^{-m} \frac{m^r}{r!} = 200 \times (0.61) \times \frac{(0.5)^r}{r!}$$ $\boxed{4 \text{ marks}}$

r	0	1	2	3	4	Total
$P(X=r)$	0.61	0.305	0.0762	0.0127	0.0016	1.0055
Expected frequency	122	61	15	2	0	200

$\boxed{7 \text{ marks}}$

Thus it gives frequencies 122, 61, 15, 2, 0 respectively for $r = 0, 1, 2, 3$ and 4. $\boxed{8 \text{ marks}}$

TEXT BOOK

I. Higher Engineering Mathematics –Dr. B.S.Grewal, 40th Edition 2007, Khanna Publishers, Delhi

II. A Text book of engineering Mathematics – N.P. Bali and Manish Goyal , 7th Edition 2007, Laxmi Publication(P) Ltd