Q.2 a. State and Prove Euler's theorem **(2+6)**

Answer:

Q.2 (a) Statement: If u is a homogeneous function of degree n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: Since u is a homogeneous function of degree n in x and y, therefore u can be written as

$$u = x^n f\left(\frac{y}{x}\right) \quad \text{——(1)}$$

Diff. (1) w.r.t. x and y partially,

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \quad \text{——(2)}$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} \quad \text{——(3)}$$

multiply (2) by x and (3) by y and add

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

Hence $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

b. If u = x + y + z, uv = y + z, uvw = z, show that **(8)**

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$$

Answer:

(b) Here $u = x+y+z$, $uv = y+z$, $uvw = z$

$\therefore x = u - uv$, $y = uv - uvw$, $z = uvw$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

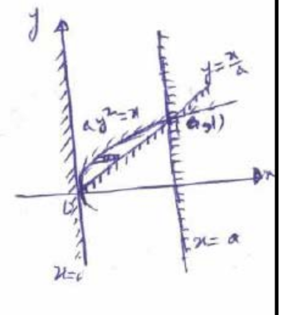
$R_1' = R_1 + R_2 + R_3$, $R_2' = R_2 + R_3$

$$\begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = u^2 v$$

Q.3 a. Change the order of integration and evaluate $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dx\,dy$ **(8)**

Answer:

Q.3 (a) Plotting the limits $x=0, x=a, y=\frac{x}{a}, y^2=\frac{x}{a}$, we get the region of integration as shown. To change order of integration, take strips $\parallel$ x-axis. Strips move $\parallel$ x-axis from $y=0$ to $y=1$ (intersection of $y=\frac{x}{a}, x=a$). Keeping ends on $x=ay^2$ and $x=ay$. Therefore changed order of integration is



$$\int^1 \int^{ay} (x^2+y^2)\,dx\,dy$$

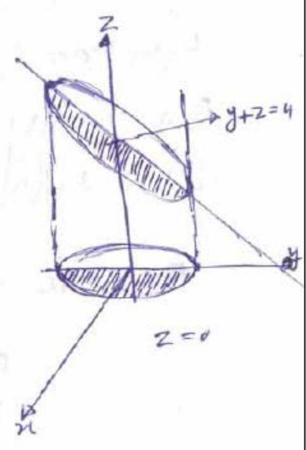
$$=\int_0^1 \left[\frac{x^3}{3}+y^2x\right]_{ay^2}^{ay} dy = \int_0^1 \left[\frac{1}{3}(a^3y^3-a^3y^6)+y^2(ay-ay^2)\right] dy$$

$$=\frac{1}{3}\left(a^3\frac{1}{4}-a^3\frac{1}{7}\right)+\left(a\frac{1}{4}-a\frac{1}{5}\right) = \frac{a^3}{28}+\frac{a}{20}$$

b. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y+z = 4$ and $z = 0$ **(8)**

Answer:

(b) Required volume is bounded by the cylinder $x^2+y^2=4$ and cut off by planes $z=0, z=4-y$.



$$\therefore \text{Reqd vol} = \int_{-2}^{+2}\int^{\sqrt{4-y^2}}\int_{z=0}^{z=4-y} dx\,dy\,dz$$

$$= 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2}\left[4x-yx\right]_0^{\sqrt{4-y^2}} dy$$

$$= 2\int_{-2}^{2}\left(4\sqrt{4-y^2}-y\sqrt{4-y^2}\right) dy \qquad = 8\int_{-2}^{2}\sqrt{4-y^2}\,dy \qquad \text{Put } y=2\sin\theta$$

$$= 16\int_{-\pi/2}^{\pi/2} 2\cos^2\theta\, d\theta = 32\times 2\times\frac{1}{2}\cdot\frac{\pi}{2} = 16\pi$$

Q.4 a. Investigate for consistency of the following equations and if possible find the solution: 4x -2y + 6z = 8, x + y – 3z = -1, 15x -3y + 9z = 21 (8)

Answer:

Q. 4(a) The given equations can be rewritten in matrix form as

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$R_1' = R_1 + 2R_2$, $R_3' = R_3 + 3R_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & -3 \\ 18 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 18 \end{bmatrix}$$

$R_3' = R_3 - 3R_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$$

Evidently rank of coeff. matrix = rank of Augmented matrix = 2 < no. of variables.

$\therefore$ Eqns are consistent and have proportionate solution.

$\therefore$ $6x = 6$ or $x = 1$

$x + y - 3z = -1$. If $z = k$, $y = -2 + 3k$ for all k's.

b. Find the eigen values and eigen vectors of the matrix. (8)

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Answer:

(b) Characteristic eqn of the given matrix is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

Roots of Characteristic eqn are 2, 3, 6 which are eigen values

Eigen Vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to eigen value $\lambda = 2$ is given by

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \xrightarrow{R_1' = R_1 - R_3,\ R_2' = R_2 + R_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$\therefore y_1 = 0$ and $x_1 - y_1 + z_1 = 0$, $x_1 = 1, y_1 = 0, z_1 = -1$

$\therefore$ Eigen Vector corresponding to eigen value $\lambda = 2$ is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Eigen Vector $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to eigen value $\lambda = 3$ is given by

$$\begin{bmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \xrightarrow{R_2' = R_2 + R_3 + R_1} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $x_2 = x_3$, $x_1 = x_2$ $\therefore x_1 = x_2 = x_3 = 1$

$\therefore$ Eigen Vector corresponding to eigen value $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Eigen Vector $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to eigen value $\lambda = 6$ is given by

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \begin{bmatrix} -4 & -2 & 0 \\ -1 & -1 & -1 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$4x_1 + 2x_2 = 0$, $x_1 + x_2 + x_3 = 0$, $2x_2 + 4x_3 = 0$; $\therefore x_1 = 1$, $x_2 = -2$, $x_3 = 1$

$\therefore$ Eigen Vector corresponding to eigen value $\lambda = 6$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Q.5 a. Use Regula-Falsi method to compute a real root of the equation x $\ell og_{10} x = 1.2$ correct to three decimal. (8)

Answer:

Q.5 (a) Here $f(x) = x\log_{10} x - 1.2 = 0$

$f(2) = 2\log_{10} 2 - 1.2 = -0.59794 = -ve$

$f(3) = 3\log_{10} 3 - 1.2 = 0.23136 = +ve$

$\therefore$ Root lies between 2 and 3, ~~let the approximate be~~ Let $x_0 = 2$, $x_1 = 3$

$\therefore$ By method of Regula-Falsi method, 1st approximation is

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 - \frac{3-2}{.23136 + .59794} \times (-.59794) = 2.7\text{[illegible]}$$

$f(2.721) = 2.721 \log_{10} 2.721 - 1.2 = -0.01709$

$\therefore$ Root lies between 2.7210 and 3. Therefore next approximation is

$$x_3 = 2.7210 - \frac{3 - 2.7210}{.23136 + .01709} \times (-.01709) = 2.7402$$

$f(2.7402) = 2.7402 \log_{10} 2.7402 - 1.2 = -.0004$

It shows that read approximation root is 2.740

b. Apply Runge-kutta method to find an approximate value of y for x = 0.2 if $\frac{dy}{dx} = x + y^2$ given that y = 1 when x = 0. Take h = 0.2 **(8)**

Answer:

(b) Here $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

By Runge-Kutta method

$k_1 = hf(x_0, y_0) = .2(0 + 1^2) = .2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = .2[0.1 + (1 + \frac{.2}{2})^2]$

$= .2[.1 + (1.1)^2] = .262$

$k_3 = hf(x_0 + \frac{h}{2}, y + \frac{k_2}{2}) = .2[(0 + .1) + (1 + \frac{.262}{2})^2]$

$= .2[.1 + (1.131)^2] = 0.2758$

$k_4 = hf(x_0 + h, y_0 + k_3) = .2[(0 + .2) + (1 + .2758)^2]$

$= .2[.2 + (1.2758)^2] = .3655$

$K = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{.2 + .424 + .5516 + .3655}{6} = 0.27$

∴ Reqd Sol. is

$y = y_0 + k = 1.2735$

Q.6 a. Use method of variation of parameters to solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$ **(8)**

Answer:

Q.6 (a) Given eqn. can be rewritten as

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \quad —(1)$$

Which is Leibnitz's equation, where $P = \frac{1}{\sqrt{x}}$, $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

∴ $I.F = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$

∴ Reqd Sol. is

$$y e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}} dx + c$$

or

$$y e^{2\sqrt{x}} = 2\sqrt{x} + c$$ Answer

b. Solve the simultaneous equation: $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cos t$ **given that x = 2 and y = 0 when t = 0.** **(8)**

Answer:

(b) Differentiate the given equation $xy = c^2$,

$$x\frac{dy}{dx} + y = 0 \quad \text{——— (1)}$$

Constant c is automatically eliminated. So eqn of orthogonal trajectories (replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$),

$$-x\frac{dx}{dy} + y = 0 \quad \text{or} \quad x\,dx - y\,dy = 0 \quad \text{— (2)}$$

Integrating (2), we get family of orthogonal system as

$$x^2 - y^2 = K$$

Q.7 a. Obtain the series solution of $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ **(8)**

Answer:

Q.7 (a) A.E. of given equation is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0$

$\therefore$ Roots of AE are 1, 1.

$\therefore$ C.F. is $\quad y = (c_1 + c_2 x)e^x$

To find P.I. by Variation of parameter method

let $y_1 = e^x \; , \; y_2 = xe^x \; , \; X = \frac{e^x}{x}$

$\therefore$ Wronskian $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$

$$\therefore \; P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^x \int \frac{xe^x}{e^{2x}} \cdot \frac{e^x}{x} dx + xe^x \int \frac{e^x}{e^{2x}} \cdot \frac{e^x}{x} dx$$

$$= -xe^x + xe^x \log x$$

Hence Complete sol. is

$$y = (c_1 + c_2 x)e^x - xe^x + xe^x \log x$$

$$= (c_1 + c_2' x)e^x + xe^x \log x \qquad \text{where } c_2' = c_2 - 1$$

b. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}}\, dx = \Pi$ **(8)**

Answer:

b. The simultaneous eqn Can be rewritten as

$$Dx + y = \sin t \quad -(1)$$
$$x + Dy = \cos t \quad -(2) \qquad \text{where } D = \frac{d}{dt}$$

Operate (1) by D and from it subtract (2), we get

$(D^2 - 1)x = 0 \quad$ whose sol. is $\quad x = c_1 e^t + c_2 e^{-t} \quad -($

Substituting for x from (3) in (1), we get

$$y = \sin t - (c_1 e^t - c_2 e^{-t}) = \sin t - c_1 e^t + c_2 e^{-t} \quad -($$

Using $x = 2, y = 0$ when $t = 0$, (3) + (4) give

$$2 = c_1 + c_2 \quad \text{and} \quad 0 = -c_1 + c_2 \; , \therefore c_1 = c_2$$

$\therefore \; c_1 = c_2 = 1$

Hence reqd. sol. is $\quad x = e^t + e^{-t} \; , \; y = \sin t - e^t + e^{-t}$

Q.8 a. Prove that $(1-x^2)\ P_n'(n) = (n+1)[x\,P_n(n) - P_{n+1}(n)]$ (8)

Answer:

Q.8 (a) Here $x=0$ is an ordinary point since coeff of $y'' \neq 0$ at $x=0$

∴ Let Series Solution be $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

Substituting these in the given eqn.

$$\left[2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 5.4a_5 x^3 + \cdots\right] + x\left[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right]$$
$$+\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right] = 0$$

Equating to zero, the coefficients of various powers of x,

$2a_2 + a_0 = 0 \quad \therefore\ a_2 = -\frac{a_0}{2}$

$3.2a_3 + a_1 + a_1 = 0 \quad \text{or } a_3 = -\frac{a_1}{3}$

$4.3a_4 + 2a_2 + a_2 = 0 \quad \text{or } a_4 = -\frac{a_2}{4} = \frac{a_0}{2\times 4}$

$5.4a_5 + 3a_3 + a_3 = 0 \quad \text{or } a_5 = -\frac{a_3}{5} = \frac{a_1}{3.5}$

$6.5a_6 + 4a_4 + a_4 = 0 \quad \text{or } a_6 = -\frac{a_4}{6} = -\frac{a_0}{2.4.6}$

$7.6a_7 + 5a_5 + a_5 = 0 \quad \text{or } a_7 = -\frac{a_5}{7} = -\frac{a_1}{3.5.7}$

etc

Substituting these, we get

$$y = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{2.4}x^4 + \frac{a_1}{3.5}x^5 - \frac{a_0}{2.4.6}x^6 - \frac{a_1}{3.5.7}x^7 + \cdots$$
$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \cdots\right)$$

b. Show that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(n)$ (8)

Answer:

(b). $\int_0^{\pi/2} \sqrt{\sin x}\, dx = \int_0^{\pi/2} \sin^{\frac12} x \cos^0 x\, dx = \frac12 \beta\left(\frac{\frac12+1}{2}, \frac{0+1}{2}\right) = \frac12 \beta\left(\frac34, \frac12\right)$ Reqd-Soln

$\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx = \int_0^{\pi/2} \sin^{-\frac12} x \cos^0 x\, dx = \frac12 \beta\left(\frac{-\frac12+1}{2}, \frac{0+1}{2}\right) = \frac12 \beta\left(\frac14, \frac12\right)$

Hence $\int_0^{\pi/2} \sqrt{\sin x}\, dx \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx = \frac14 \beta\left(\frac34, \frac12\right) \beta\left(\frac14, \frac12\right) = \frac14 \frac{\Gamma\frac34 \Gamma\frac12}{\Gamma\frac54} \times \frac{\Gamma\frac14}{\Gamma\frac34}$

$= \frac14 \frac{\sqrt{\pi}}{\frac14 \Gamma\frac14} \Gamma\frac14 \sqrt{\pi} = \pi$

Q.9 (For Current Scheme students i.e. AE51/AC51/AT51)

a. Solve the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ **(8)**

Answer:

Q. 9 (a) We know from recurrence formulae that

$(1+x^2) P_n'(x) = n\left[P_{n-1}(x) - x P_n(x)\right]$ —— (1)

and $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$ —— (2)

$(1-x^2) P_n'(x) = (2n+1) x P_n(x) - (n+1) P_{n+1}(x) - n x P_n(x)$

$= (n+1) x P_n(x) - (n+1) P_{n+1}(x)$

$= (n+1)\left[x P_n(x) - P_{n+1}(x)\right] = RHS$

b) We k...

b. Find the orthogonal trajectories of the family of hyperbolas xy = C^2 **(8)**

Answer:

= RHS.

(b) We know

$$e^{\frac{x}{2}(t-\frac{1}{t})} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$$

$$= \left[1 + \frac{xt}{2} + \frac{1}{\lfloor 2}\frac{x^2t^2}{2^2} + \frac{1}{\lfloor 3}\frac{x^3t^3}{2^3} + \cdots\right]\left[1 - \frac{x}{2t} + \frac{1}{\lfloor 2}\left(\frac{x}{2t}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

Multiply the two series term by term and collect coefficients of all powers of t.

Coeff of t^0 = term independent of t $= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(\lfloor 2)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(\lfloor 3)^2}\left(\frac{x}{2}\right)^6 + \cdots$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(\lfloor k)^2}\left(\frac{x}{2}\right)^{2k} = J_0(x)$$

Coeff of $t^1 = \frac{x}{2} - \frac{1}{\lfloor 2}\left(\frac{x}{2}\right)^3 + \frac{1}{\lfloor 3 \lfloor 2}\left(\frac{x}{2}\right)^5 - \frac{1}{\lfloor 4 \lfloor 3}\left(\frac{x}{2}\right)^7 + \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{\lfloor k \lfloor k+1}\left(\frac{x}{2}\right)^{1+2k} = J_1(x)$

Coeff of $t^n = \frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n - \frac{1}{\lfloor n+1}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{\lfloor n+2 \lfloor 2}\left(\frac{x}{2}\right)^{n+4} - \cdots$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{\lfloor (n+k) \lfloor k}\left(\frac{x}{2}\right)^{n+2k} = J_n(x)$$

Similarly we can collect coeffts of various powers of $\left(-\frac{1}{t}\right)$. We find that coefficient of $\left(-\frac{1}{t}\right)^n$ is same as $J_n(x)$ or coeff of t^{-n} is $(-1)^n J_n(x) = J_{-n}(x)$.

Hence we get

$$e^{\frac{x}{2}(t-\frac{1}{t})} = J_0(x) + J_1(x)\,t + J_2(x)\,t^2 + \cdots + J_n(x)\,t^n + \cdots$$
$$+ J_{-1}(x)\,t^{-1} + J_{-2}\,t^{-2} + \cdots + J_{-n}(x)\,t^{-n} + \cdots$$
$$= \sum_{n=-\infty}^{\infty} J_n(x)\,t^n.$$

Q.9 (For New Scheme students i.e. AE101/AC101/AT101)

a. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ (8)

Answer:

(102/15

9(a) Let $f(x) = \dfrac{e^{-ax}}{x}$. The Fourier Sine transform

$$F_s(f(x)) = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = F(s) \quad (2)$$

Diff both sides w.r.to s

$$\frac{d}{ds}(F(s)) = \int_0^\infty \frac{x e^{-ax}\cos sx}{x}\,dx = \int_0^\infty e^{-ax}\cos sx\,dx \quad (2)$$

$$= \frac{a}{s^2+a^2} \quad (2).$$

∴ Integrating $F(s) = \int_0^s \frac{a}{s^2+a^2}\,ds = \tan^{-1}\frac{s}{a} + c$ (1)

$F(s) = 0$ when $s = 0$, $\Rightarrow c = 0$ ∴ $F(s) = \tan^{-1}\frac{s}{a}$ (1).

b. Find the z-transform of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ **(8)**

Answer:

9(b) $$Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = Z\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]$$

$$= \cos\frac{\pi}{4}\,Z\left(\cos n\frac{\pi}{2}\right) - \sin\frac{\pi}{4}\,Z\left(\sin n\frac{\pi}{2}\right) \quad (2)$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\frac{\pi}{2}+1} - \frac{z\sin\pi/2}{z^2-2z\cos\frac{\pi}{2}+1}\right\} \quad (2+2)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right) = \frac{1}{\sqrt{2}}\,\frac{z(z-1)}{(z^2+1)} \quad (2).$$

Marking Scheme
for
Engg. maths-I (AE51)

Q.N

2(a) 2 marks for statement, 2 marks for (1), 1 mark each for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and 8 for full correct

(b) 2 marks for definition $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$, $\frac{1}{2}$ mark for each partial derivative + 8 for full correct

3(a) 4 marks for change of order of integration and 4 for evaluation of integral

(b) 2 for correct formula $Vol = \iiint dx\,dy\,dz$, 2 marks for correct limits, 8 for full correct

4(a) 4 marks for consistency and 4 for solution.

(b) 2 marks characteristic eqn $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$, 2 more marks for eigen values + 4 for eigen vector

5(a) 2 marks for showing that root lies between 2 and 3, 2 marks for correct formula, 4 more for correct evaluation

(b) 1 mark for each k_1, k_2, k_3, k_4, 2 for K + 8 for full $y(0.2)$

6(a) 2 marks for writing in Leibnitz form (1), 2 marks for I.F., 8 for full correct.

(b) 2 for eqn. (1), 2 for replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, 8 for full correct

7(a) 2 marks for C.F., 2 marks for Wronskian, 2 marks for correct formula for P.I., and 8 for full correct

(b) 2 marks for writing eqn in (1)+(2) form, 4 marks for getting $(D^2-1)x = 0$, 4 for itself $x = c_1e^t + c_2e^{-t}$. 2 for getting $y = \sin t - c_1e^t + c_2e^{-t}$, 2 more for getting $c_1 + c_2$ + correct Answer

Q. 8(a) 2 marks for changing $y = a_0 + a_1x + a_2x^2$, $y' = a_1 + 2a_2x + \cdots$, $y'' = 2a_2 + 3\cdot2a_3x + \cdots$.
4 marks correct comparing of like powers of x getting $a_2 = -\frac{a_0}{2}$, $a_3 = -\frac{a_1}{3}$, etc
8 for full correct

(b) 2 marks $\int_0^{\pi/2}\sqrt{\sin x}\,dx = \frac{1}{2}\beta(\frac{3}{4},\frac{1}{2})$, 2 marks for $\int_0^{\pi/2}\frac{1}{\sqrt{\sin x}}dx = \frac{1}{2}\beta(\frac{1}{4},\frac{1}{2})$ and 8 for full correct

Q9 (a) 2 marks for writing (1), 2 marks for (2), 2 marks for eliminating $P_{n-1}(x)$, 8 for full correct

(b) 2 marks for correct expansion of $e^{\frac{xt}{2}}$, $e^{-\frac{x}{2t}}$, 4 marks for collecting coefficients of $t^0, t^1, t^2, \ldots, t^{-1}, t^{-2}, \ldots$ and expressing in $J_0, J_1, \ldots, J_{-1}, J_{-2}, \ldots$
8 for full correct.

<u>TEXT BOOK</u>

I. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi

II. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd