Q.2 a. Evaluate $\lim\limits_{x\to 0}\left(\frac{\cos Ax - \cos Bx}{x^2}\right)$

Q.2.a. Soln:

We have, $\underset{x\to 0}{\text{Limit}}\left(\frac{\cos Ax - \cos Bx}{x^2}\right)$ (form $\frac{0}{0}$)

$= \underset{x\to 0}{Lt} \frac{2\sin\left(\frac{A+B}{2}\right)x \cdot \sin\left(\frac{B-A}{2}\right)x}{x^2}$ $\left[\because \cos C - \cos D \to 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}\right]$

2 Marks

$= 2\underset{x\to 0}{Lt}\left\{\frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x}\times\left(\frac{A+B}{2}\right)\frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x}\times\left(\frac{B-A}{2}\right)\right\}$

2 Marks

$= 2\left(\frac{B+A}{2}\right)\left(\frac{B-A}{2}\right)\underset{x\to 0}{Lt}\left\{\frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x}\right\}\times\left\{\underset{x\to 0}{Lt}\frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x}\right\}$

2 Marks

$= \left(\frac{B-A^2}{2}\right)(1)(1) = \frac{B^2-A^2}{2}$ Ans.

b. Expand e^x in powers of (x + 3).

Q.2.b. Soln: $f(x) = e^x$

by Taylor's Theorem,

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$ — 2 Marks

Here, $h = x+3$

$a+h = x$ or $a + (x+3) = x$, $a = -3$

Putting the values of a, h in Taylor's Theorem, we get

$f(-3+x+3) = f(-3) + (x+3)f'(-3) + \frac{(x+3)^2}{2!}f''(-3) + \frac{(x+3)^3}{3!}f'''(-3) + \cdots$

$e^x = e^{-3} + (x+3)e^{-3} + \frac{1}{2!}(x+3)^2e^{-3} + \frac{(x+3)^3}{3!}e^{-3} + \cdots$ — 2 Marks

$e^x = e^{-3}\left[1 + (x+3) + \frac{(x+3)^2}{2!} + \frac{(x+3)^3}{3!} + \cdots\right]$

$e^x = \frac{1}{20}\left[1 + (x+3) + \frac{(x+3)^2}{2!} + \frac{(x+3)^3}{3!} + \cdots\right]$ Ans

Q.3 a. Evaluate $\int_0^2 \sqrt{\frac{2+x}{2-x}}\,dx$

Q.3.a. Soln. $\int_0^2 \sqrt{\frac{2+x}{2-x}}\,dx$

Rationalising the numerator,

$$= \int_0^2 \frac{2+x}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx + \int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx \quad \text{— 2 Marks}$$

$$= \left[2\sin^{-1}\frac{x}{2}\right]_0^2 - \left[\sqrt{4-x^2}\right]_0^2 \quad \text{— 4 Marks}$$

$$= 2\left[\sin^{-1}1 - \sin^{-1}0\right] - \left[\sqrt{4-4} - 2\right]$$

$$= 2\left[\frac{\pi}{2}\right] + 2 = \pi + 2 \qquad \text{Ans } (\pi+2)$$

b. Find the length of the curve $x = a\cos^3\theta, y = a\sin^3\theta$ **, in the first quadrant.**

Q.3.b. Soln.

Here $\frac{dx}{d\theta} = -3a\cos^2\theta \cdot \sin\theta$

$\frac{dy}{d\theta} = 3a\sin^2\theta \cdot \cos\theta$

and θ varies from 0 to $\frac{\pi}{2}$ for the first quadrant.

Required length of the curve $= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ — 2 Marks

$$= \int_0^{\pi/2} \sqrt{9a^2\cos^4\theta \sin^2\theta + 9a^2\sin^4\theta \cdot \cos^2\theta}\, d\theta$$

$$= 3a\int_0^{\pi/2} \sin\theta \cdot \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}\, \cdot d\theta$$

4 Marks

$$= 3a\int_0^{\pi/2} \sin\theta \cdot \cos\theta\, d\theta$$

$$= 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3a}{2} \text{ Ans}$$ 8 Marks

Q.4 a. Separate into real and imaginary part of sec (x + iy)

Q.4.a. Soln.

We have, $\sec(x+iy) = \frac{1}{\cos(x+iy)}$

$\Rightarrow \sec(x+iy) = \frac{2\cos(x-iy)}{2\cos(x+iy)\cdot\cos(x-iy)}$ — 2 Marks

$\Rightarrow \sec(x+iy) = \frac{2[\cos x.\cos(iy) + \sin x.\sin(iy)]}{\cos 2x + \cos(i2y)}$

$\Rightarrow \sec(x+iy) = \frac{2\cos x.\cosh y + 2i\sin x.\sinh y}{\cos 2x + \cosh 2y}$ 4 Marks

$\Rightarrow \sec(x+iy) = \frac{2\cos x.\cosh y}{\cos 2x + \cosh y} + i\,\frac{2\sin x.\sinh y}{\cos 2x + \cosh 2y}$

Ans $\left[\begin{array}{l} \therefore \text{Real part of } \sec(x+iy) = \frac{2\cos x.\cosh y}{\cos 2x + \cosh 2y} \\ \text{and Imaginary part of } \sec(x+iy) = \frac{2\sin x.\sinh y}{\cos 2x + \cosh 2y} \end{array}\right.$ 8 Marks

b. If two impedance $Z_1 = 50\ \angle -40^0$ and $Z_2 = 70\angle 30^0$ are connected in series, find the total impedance in polar form.

Q.4.b. Soln:

We have,

$Z_1 = 50 \angle -40^\circ = 50\{\cos(-40^\circ) + i\sin(-40^\circ)\}$

$\Rightarrow Z_1 = 50(\cos 40^\circ - i\sin 40^\circ) = 50(0.7660 - i\,0.6428)$

$= 38.3 - 32.14i$ — 2 Marks

and $Z_2 = 70\angle 30^\circ = 70(\cos 30^\circ + i\sin 30^\circ) = 70(0.8660 + 0.5i)$

$= 60.62 + 35i$

Since Z_1 and Z_2 are connected in series,

$\therefore$ Total impedance $Z = Z_1 + Z_2$

$= (38.3 - 32.14i) + (60.62 + 35i)$

$= 98.92 + 2.86i$ — 4 Marks

$\Rightarrow |Z| = \sqrt{(98.92)^2 + (2.86)^2} = 98.94$

The argument θ of Z is given by

$\tan\theta = \dfrac{2.86}{98.92} \Rightarrow \theta = 1.656^\circ$ — 1 Mark

Hence, the polar form of Z is $98.94\angle 1.656$

or $98.94(\cos\theta + \sin\theta)$, where $\theta = 1.656^\circ$

Q.5 a. If $\vec{a} = i + 2j - 3k$ and $\vec{b} = 3i - j + 2k$, show that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular to each other. Also find the angle between $2\vec{a}+\vec{b}$ and $\vec{a}+2\vec{b}$.

Q.5.a. Soln:

$\vec{a}+\vec{b} = 4i+j-k$, and $\vec{a}-\vec{b} = -2i+3j-5k$

Now $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = (4i+j-k)\cdot(-2i+3j-5k)$

$= -8+3+5 = 0$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$ — 4 Marks

Hence $(\vec{a}+\vec{b})$ is $\perp$ to $(\vec{a}-\vec{b})$

Let $\vec{A} = 2\vec{a}+\vec{b} = (2i+4j-6k)+(3i-j+2k)$

$\Rightarrow \vec{A} = 5i+3j-4k$

and $\vec{B} = \vec{a}+2\vec{b} = (i+2j-3k)+(6i-2j+4k)$

$\Rightarrow \vec{B} = (7i-0\cdot j+k)$

Now angle between $\vec{A}$ and $\vec{B}$ is given as,

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$ — 2 Marks

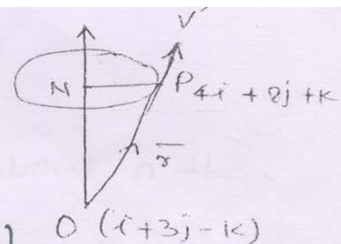
$$\cos\theta = \frac{(5i+3j-4k)\cdot(7i+0\cdot j+k)}{\sqrt{25+9+16}\cdot\sqrt{49+0+1}}$$

$$\cos\theta = \frac{35-4}{\sqrt{50}\times\sqrt{50}} = \frac{31}{50} \Rightarrow \theta = \cos^{-1}\left(\frac{31}{50}\right)$$

So angle between $(2\vec{a}+\vec{b})$ and $(\vec{a}+2\vec{b})$ is $\cos^{-1}\left(\frac{31}{50}\right)$. Ans — 4 Marks

b. **A rigid body is spinning with an angular velocity of 27 radian / sec about an axis parallel to** $2\hat{i}+\hat{j}-2\hat{k}$ **passing through the point** $\hat{i}+3\hat{j}-\hat{k}$**. Find the velocity of the point whose position vector is** $4i+8j+k$**.**

Q.3.a. Soln.



Let $\vec{w}$ be the angular of the body rotating about an axis parallel to the vector $2i+j-2k$.

then, $\vec{w} = 27 \cdot \frac{2i+j-2k}{\sqrt{4+1+4}}$ — 2 Marks

$= 18i + 9j - 18k$

Let $\vec{r} = \vec{OP}$ = P.v. of P − P.v. of O

$\vec{r} = (4i + 8j + k) - (i + 3j - k)$

$\vec{r} = 3i + 5j + 2k$ —— 2 Marks

Let $\vec{v}$ be the velocity of the particle at P.

Then, $\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} i & j & k \\ 18 & 9 & -18 \\ 3 & 5 & 2 \end{vmatrix}$

$\vec{v} = 108i - 90j + 63k$ — 4 Marks

$= 9(12i - 10j + 7k)$ Ans.

Q.6 a. Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x.\sin x$

Q.6.a. Soln:

$$(D^2-2D+1)y = x e^x \sin x \qquad \left(\because D \equiv \frac{d}{dx}\right)$$

A.E. $D^2-2D+1=0$

or $(D-1)^2=0$, or $D=+1, +1$

C.F. $=(c_1+c_2x)e^x$ — 3 Marks

$$\text{P.I.} = \frac{1}{(D-1)^2}\cdot x e^x \sin x = e^x \cdot \frac{1}{(D+1-1)^2}\cdot x \sin x$$

$$= e^x \cdot \frac{1}{D^2} x \sin x = e^x \cdot \frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x[-x\sin x - \cos x - \cos x]$$

$$= -e^x[x\sin x + 2\cos x]$$ — 5 Marks

$$y = (c_1+c_2x)e^x - e^x[x\sin x + 2\cos x] \quad \text{Ans.}$$

b. The deflection of a strut of length ℓ with one end (x = 0) build-in and the other supported and subjected to end thrust P, satisfies the equation

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(\ell - x)$$

Prove that the deflection curve is $y = \frac{R}{P}\left(\frac{\sin ax}{a} - \ell\cos ax + \ell - x\right)$**, where** $a\ell = \tan a\ell$

Ans Ex14.13, page 566 of B S Grewal

Q.7 a. Obtain the Fourier series to represent the function f(x) = $|x|$ **for** $-\pi < x < \pi$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

Q.7.a. Soln.

We have $f(x) = |-x| = |x| = f(x)$

So, $f(x)$ is an even function. Therefore, the fourier series consists of cosine terms only and is given by,

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, — (1) where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$

2 marks

Determination of a_0, we have,

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$

$\Rightarrow a_0 = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \frac{2}{\pi} \times \frac{\pi^2}{2} = \pi$ — 2M

Determination of a_n: we have,

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$

$\Rightarrow a_n = \frac{2}{\pi}\left[x\frac{\sin x}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$

$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1)$ — 4 marks

$\Rightarrow a_n = \frac{2}{n^2\pi}\{(-1)^n - 1\}$

$\Rightarrow a_n = \begin{cases} -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

Substituting the values of a_0, a_n, b_n in (I)
we obtain,

$$\frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\} \quad \text{(II)}$$

as the Fourier Series of $f(x) = |x|$

Since $f(x)$ is continuous on $[-\pi, \pi]$

$$\therefore\ f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\} \quad \text{(III)}$$

The sum of the series at $x = 0$, is $f(0) = |0| = 0$

Putting $x = 0$ in (III), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

2 Marks

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Ans

b. Find the Fourier series for the function f(x) = x in the interval $[-\pi, \pi]$

Q.7.b Soln.

Clearly, $f(x) = x$ is an odd function. Therefore, Fourier Series for $f(x)$ is purely a Sine Series given by

2 Mark

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{(I)}, \quad \text{where, } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Since $f(x) = x$ is continuous for all $x \in (-\pi, \pi)$

$\therefore$ Fourier Series of $f(x)$ converges to $f(x)$ for all $x \in (-\pi, \pi)$. Hence

$$x = \sum_{n=1}^{\infty} b_n \sin nx$$

Computation of b_n: We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(-\pi\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - 0\right]$$

$$\Rightarrow b_n = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

4 Marks

Substituting the value of b_n in (I), we obtain

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

or $2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right\}$

as the Fourier Series of $f(x)$.

Q.8 a. Find the Laplace transform of $e^{-t}(3\sin t - \cos^2 t)$

Q.8.a. Soln.

We have, $L\{3\sin t - \cos^2 t\} = L\left\{3\sin t - \frac{(1+\cos 2t)}{2}\right\}$

$$= \frac{1}{2}L\{6\sin t - \cos 2t - 1\}$$

$$= \frac{1}{2}[6L\{\sin t\} - L\{\cos 2t\} - L\{1\}]$$ 4 Marks

$$= \frac{1}{2}\left\{\frac{6}{s^2+1} - \frac{s}{s^2+4} - \frac{1}{s}\right\} = \frac{3}{s^2+1} - \frac{s^2+2}{s(s^2+4)}$$

$$\therefore L\{e^{-t}(3\sin t - \cos^2 t)\} = \frac{3}{(s+1)^2+1} - \frac{(s+1)^2+2}{(s+1)\{(s+1)^2+4\}}$$

$$\Rightarrow L\{e^{-t}(3\sin t - \cos^2 t)\} = \frac{3}{s^2+2s+2} - \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}$$

Ans

b. Find the Laplace transform of $t\sin^2 t$

Q.8.b. Soln. 8 Marks

We have, $L\{\sin^2 t\} = \frac{2}{s(s^2+4)}$ — 2 Marks

$\therefore L\{t\sin^2 t\} = -\frac{d}{ds}\left\{\frac{2}{s(s^2+4)}\right\}$ — 2 Marks

$$\Rightarrow L\{t\sin^2 t\} = -2\frac{d}{ds}\{s^{-1}(s^2+4)^{-1}\}$$

$$= -2\left\{-s^{-2}(s^2+4)^{-1} - s^{-1}(s^2+4)^{-2}\frac{d}{ds}(s^2+4)\right\}$$

$$\Rightarrow L\{t\sin^2 t\} = -2\left\{-\frac{1}{s^2(s^2+4)} - \frac{1}{s(s^2+4)^2}\times 2s\right\}$$

$$= \frac{2(s^2+2s+4)}{s^2(s^2+4)^2}$$ Ans — 8 Marks

Q.9 a. Find $L^{-1}\left\{\frac{2s^2-4}{(s-2)(s+\ell)(s-3)}\right\}$

Q.9.a. Soln:

Let $\dfrac{2s^2-4}{(s-2)(s+1)(s-3)} \equiv \dfrac{A}{s-2} + \dfrac{B}{s+1} + \dfrac{C}{s-3}$ — 2

Multiplying both sides by $(s-2)(s+1)(s-3)$, we get Marks

$$2s^2-4 \equiv A(s+1)(s-3) + B(s-2)(s-3) + C(s-2)(s+1)$$

This is an identity in s and so it holds for all values of s. Putting $s = 2, -1, 3$, we get,

$$A = -\frac{4}{3}, \quad B = -\frac{1}{6}, \quad C = \frac{7}{2}$$

Thus we have,

$$\frac{2s^2-4}{(s-2)(s+1)(s-3)} \equiv \frac{-\frac{4}{3}}{s-2} + \frac{-\frac{1}{6}}{s+1} + \frac{\frac{7}{2}}{s-3}$$ — 2 Marks

or $$\frac{2s^2-4}{(s-2)(s+1)(s-3)} \equiv -\frac{4}{3}\times\frac{1}{s-2} - \frac{1}{6}\cdot\frac{1}{s+1} + \frac{7}{2}\cdot\frac{1}{s-3}$$

$$\Rightarrow L^{-1}\left\{\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right\} = -\frac{4}{3}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right\} = -\frac{4}{3}e^{2t} - \frac{1}{6}e^{-t} + \frac{7}{2}e^{3t}$$ Ans — 4 Marks

b. Find $L^{-1}\left\{\dfrac{\ell}{s^3(s^2+\ell)}\right\}$

Q.9.b. Soln.

We have,

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \quad \text{— 2 Marks}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2+1}\right\} dt \quad \text{— 2 Marks}$$

$$= \int_0^t \sin t \, dt = 1 - \cos t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} dt \quad \text{— 2 Marks}$$

$$= \int_0^t (1-\cos t)\, dt = t - \sin t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} dt$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (t - \sin t)\, dt \quad \text{— 2 Marks}$$

$$= \frac{t^2}{2} + \cos t - 1 \quad \text{Ans}$$

Text book

1. **Engineering mathematics –Dr. B.S.Grewal, 12th edition 2007, Khanna publishers, Delhi**
2. **Engineering Mathematics – H.K.Dass, S. Chand and Company Ltd, 13th Revised Edition 2007, New Delhi**