Q.2a. State and prove Leibnitz theorem for the n^th derivative of the product of two function.

Q.2 (a) Statement:

If u and v are two functions of x, then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + {}^nC_n u v_n$$

Proof: Proof is by induction

By actual ~~multiplication~~ differentiation

$$\frac{d}{dx}(uv) = (uv)_1 = u_1 v + u v_1 = {}^1C_0 u_1 v + {}^1C_1 u v_1$$

$$\frac{d^2}{dx^2}(uv) = (uv)_2 = (u_1 v + u v_1)_1 = u_2 v + u_1 v_1 + u v_2 + u_1 v_1 = u_2 v + 2u_1 v_1 + u v_2$$

$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$$

∴ Theorem is true for $n=2$.

$$\frac{d^3}{dx^3}(uv) = (uv)_3 = (u_2 v + 2u_1 v_1 + u v_2)_1 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + u v_3$$

$$= {}^3C_0 u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 u v_3$$

∴ Theorem is true for $n=3$

Let the theorem be true for $n = m$ (say), so that

$$\frac{d^m}{dx^m}(uv) = (uv)_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots + {}^mC_{m-1} u_1 v_{m-1} + {}^mC_m u v_m$$

Differentiating

$$\frac{d^{m+1}}{dx^{m+1}}(uv) = (uv)_{m+1} = {}^mC_0 u_{m+1} v + {}^mC_0 u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2 + \dots + {}^mC_{m-1} u_2 v_{m-1}$$
$$+ {}^mC_{m-1} u_1 v_m + {}^mC_m u_1 v_m + {}^mC_m u v_{m+1}$$

$$= {}^mC_0 u_{m+1} v + ({}^mC_0 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2 + \dots + ({}^mC_{m-1} + {}^mC_m) u_1 v_m + {}^mC_m u v_{m+1}$$

$$= {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \dots + {}^{m+1}C_m u_1 v_m + {}^{m+1}C_{m+1} u v_{m+1}$$

$(\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r)$

This shows that if theorem is true for $n = m$, it is also true for $n = m+1$, i.e. the next higher value of m. We have seen above that theorem is true for $n = 1, 2, 3$, therefore it is true for next higher value $n = 3+1 = 4$ and $n = 4+1 = 5$ and so on. Hence theorem is true for ~~all~~ integral value of n.

(b) Here

$y = f(x) = (x \dots$

b.Find the points at which the function f(x) = (x-1) (x-2) (x-3) has a maximum and minimum values.

(b) Here

$y = f(x) = (x-1)(x-2)(x-3) \quad \text{—(1)}$

$\therefore \dfrac{dy}{dx} = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2) = 3x^2 - 12x + 11$

For maxima and minima, $\dfrac{dy}{dx} = 0$, i.e. $3x^2 - 12x + 11 = 0$, $\therefore x = \dfrac{6 \pm \sqrt{3}}{3}$

$\therefore$ the given function has maxima and minima at $x = \dfrac{6 \pm \sqrt{3}}{3}$

Also $\dfrac{d^2y}{dx^2} = 6x - 12$

The condition for the funct to be max. is $\dfrac{d^2y}{dx^2} = -ve$ at that pt.

and condition for min. is $\dfrac{d^2y}{dx^2} = +ve$ at that pt.

At $x = \dfrac{6+\sqrt{3}}{3}$, $\dfrac{d^2y}{dx^2} = 6\left[\dfrac{6+\sqrt{3}}{3}\right] - 12 = 2\sqrt{3} = +ve$, $\therefore f(x)$ is min

At $x = \dfrac{6-\sqrt{3}}{3}$, $\dfrac{d^2y}{dx^2} = 6\left[\dfrac{6-\sqrt{3}}{3}\right] - 12 = -2\sqrt{3} = -ve$, $\therefore f(x)$ is max

Q.3a. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

3(a). Here

$$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{---(1)}$$

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$$=\int_0^{\pi/2}\frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\cos\sqrt{\frac{\pi}{2}-x}}\,dx,\quad \text{using } \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{---(2)}$$

Adding (1) and (2),

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$$2I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx+\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2}$$

Hence given integral $= I=\frac{\pi}{4}$

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b.Integrate $\int\frac{x^2-3x+4}{(x-2)(x+2)(x+4)}dx$

(b) By partial fraction,

$$\frac{x^2-3x+4}{(x-2)(x+2)(x+4)}=\frac{1}{12(x-2)}-\frac{7}{4(x+2)}+\frac{8}{3(x+4)}$$

Integrating,

$$\int\frac{x^2-3x+4}{(x-2)(x+2)(x+4)}=\frac{1}{12}\int\frac{dx}{x-2}-\frac{7}{4}\int\frac{1}{x+2}dx+\frac{8}{3}\int\frac{1}{x+4}dx$$

$$=\frac{1}{12}\log(x-2)-\frac{7}{4}\log(x+2)+\frac{8}{3}\log(x+4)$$

Q.4 a. Compute the inverse of $\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$

Q.4 (a) By definition $A^{-1} = \frac{\text{Adj } A}{|A|}$, where Adj A is transpose of matrix of cofactor of A.

Here $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, $\therefore |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} \overset{R_1' = R_1 - R_2}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 1$

$\therefore A^{-1} = \text{Adj. } A$ = transpose of matrix of cofactors

$= \text{transpose of } \begin{bmatrix} +(-3+4) & -(2\times1-0) & +(2\times-1-0) \\ -(-3+4) & +(3-0) & -(-3-0) \\ +(-12+12) & -(12-8) & +(-9+6) \end{bmatrix}$ (replacing all elements by their cofactors)

$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

b.Show that the equations $2x + y + 2z = 1$
$x + 2y - z = 2$
$5x + 4y + 3z = 4$
are consistent and solve them.

(b). Eqns can be written in matrix form

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

or $R_3' = R_3 - 2R_1 - R_2$, $R_2' = R_2 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 2 \\ -3 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Evidently rank of coeff matrix = Rank of Augmented matrix = 2 < no. of Variables

$\therefore$ Given eqns are consistent and have infinite solns which are given by

$$2x + y + 2z = 1$$
and $$-3x - 5z = 0$$

$\therefore$ If $z = k$, $x = -\frac{5}{3}k$, $y = 1 - 2k + \frac{10}{3}k = 1 + \frac{4}{3}k$

Q.5a. Solve the differential equation:

$$\frac{dy}{dx} = \frac{y}{x} - \sqrt{\left(\frac{y^2}{x^2} - 1\right)}$$

Q5(a) Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{vx}{x} - \sqrt{\frac{v^2x^2}{x^2} - 1}$

or $\frac{dv}{\sqrt{v^2-1}} = -\frac{dx}{x}$

Integrating $\log\left[v + \sqrt{v^2-1}\right] = -\log x + C$

or $x\left[\frac{y}{x} + \sqrt{\frac{y^2}{x^2} - 1}\right] = \text{const}$ OR $y + \sqrt{y^2 - x^2} =$

5. b. Solve the differential equation:

e^{-y} **sec²y dy = dx + x dy**

(b) Given eqn. can be rewritten as

$$\frac{dx}{dy} = -x + e^{-y}\sec^2 y$$

OR $\frac{dx}{dy} + x = e^{-y}\sec^2 y$

It is linear diff eqn in x and its integrating factor is

$$I.F. = e^{\int dy} = e^{y}$$

$\therefore$ Soln given by $x \cdot e^{y} = \int e^{y} \cdot e^{-y}\sec^2 y\, dy + C$

OR $x e^{y} = \tan y + C$

Q.6a. If S_1, S_2, S_3 be the sums of n, 2n, 3n terms respectively of an A.P. show that $S_3 = 3(S_2 - S_1)$

Q6(a) Let A.P. be $a, a+d, a+2d, a+3d, \ldots$

$$\therefore S_1 = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_2 = \frac{2n}{2}\left[2a+(2n-1)d\right]$$

$$S_3 = \frac{3n}{2}\left[2a+(3n-1)d\right]$$

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$$\therefore S_2 - S_1 = \frac{2n}{2}\left[2a+(2n-1)d\right] - \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[2a+(4n-2-n+1)d\right]$$

$$= \frac{n}{2}\left[2a+(3n-1)d\right]$$

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Evidently $S_3 = 3(S_2 - S_1)$

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b.The sum of first and second terms of a G.P. is $\frac{5}{4}$ and the sum of the fourth and fifth terms is 80. Find the first term and the common ratio.

(b) Let the G.P. be $a, ar, ar^2, ar^3, ar^4, \ldots$

$\therefore$ as per given conditions

$$a + ar = \frac{5}{4} \quad \text{---(1)}$$

$$ar^3 + ar^4 = 80 \quad \text{---(2)}$$

Dividing (2) by (1), $r^3 = 64$ or $r = 4$

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using this value in (1) $a = \frac{1}{4}$

Hence 1st term $= \frac{1}{4}$ and c.r. $= 4$

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Q.7 a. If A + B + C = π, show that sin A + sin B + sin C = 4 $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Q7(a) $\sin A + \sin B + \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\sin\frac{\pi - C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\right]$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$

$= 2\cos\frac{C}{2}\left[2\cos\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\cos\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right]$

$= 4\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

Hence proved.

b.In any triangle ABC, prove that tan $\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(b) By law of sines,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\therefore \frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}$

$= \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \tan\frac{B-C}{2}\cdot\cot\frac{B+C}{2}$

$= \tan\frac{B-C}{2}\cot\frac{\pi - A}{2} = \tan\frac{B-C}{2}\tan\frac{A}{2}$

Hence $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

Q.8 a. Find the equation of the line through the point of intersection of 5x-3y = 1 & 2x + 3y = 23 and perpendicular to the line whose equation is 5x-3y = 1.

Q 8 (a) Any line thro' the point of intersection of $5x-3y-1=0$ and $2x+3y-23=0$ is

$$5x-3y-1+K(2x+3y-23)=0$$

or $$(5+2K)x+(3K-3)y-1-23K=0 \quad \text{——} (1)$$

This line is perpendicular to $5x-3y=1$ —— (2)

∴ Prod. of Slopes of (1) and (2) is −1.

i.e. $$\left[-\frac{5+2K}{3K-3}\right]\cdot\left[\frac{5}{3}\right]=-1$$

or $K=-34$

Hence the reqd st. line (1) becomes

$$(5-2\times34)x+(-3\times34-3)y-1+23\times34=0$$

or $$63x+105y-781=0$$

Ans.

b.If p and p' be the perpendiculars from the origin upon the straight lines whose equations are

$$\mathbf{x \sec\theta + y \operatorname{cosec}\theta = a}$$

and $$\mathbf{x \cos\theta - y \sin\theta = a \cos 2\theta}$$

Prove that $4p^2+(p')^2=a^2$

(b) Here p and p' are perpendiculars from origin on lines $x\sec\theta+y\operatorname{cosec}\theta-a=0$ and $x\cos\theta-y\sin\theta-a\cos2\theta=0$

∴ $$p=\frac{-a}{\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}}=\frac{-a\sin\theta\cos\theta}{\sqrt{\sin^2\theta+\cos^2\theta}}=-a\sin\theta\cos\theta$$

$$p'=\frac{-a\cos2\theta}{\sqrt{\cos^2\theta+\sin^2\theta}}=-a\cos2\theta$$

∴ $$4p^2+p'^2=4a^2\sin^2\theta\cos^2\theta+a^2\cos^2 2\theta$$

$$=a^2\sin^2 2\theta+a^2\cos^2 2\theta=a^2$$

Ans

Q.9 a. Find the vertex, focus, axis and the directrix of the parabola

$$y^2 = x + 2y - 2$$

Q.9 (a) Eqn of parabola can be rewritten as

$$y^2 - 2y + 1 = x - 2 + 1$$

or

$$(y-1)^2 = x - 1 \quad \text{or} \quad Y^2 = X \quad \text{where } X = x-1,\ Y = y-1$$

Which is parabola in X, Y system

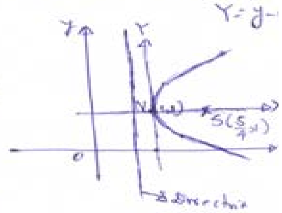
$\therefore$ Vertex $= X = 0,\ x - 1 = 0$; $Y = 0,\ y - 1 = 0$; ie $x = 1,\ y = 1$

Axis of Parabola is $Y = 0$ i.e. $y - 1 = 0$ or $y = 1$

L.R $= 4a = 1$, $\therefore a = \frac{1}{4}$

Focus is $S\left(1 + \frac{1}{4}, 1\right) = S\left(\frac{5}{4}, 1\right)$

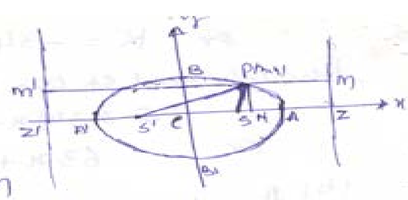
Directrix is $X = -a$ or $x - 1 = -\frac{1}{4}$ or $x = \frac{3}{4}$



b.Show that the sum of the focal distances of any point on an ellipse is constant and equal to the major axis.

(b) Let S and S′ be the foci and ZM and Z′M′ be the directrices of the ellipse.

Let $P(x, y)$ be any pt on the ellipse. Join PS, PS′. Drop ⊥s PM on ZM and PM′ on Z′M′, PN on x-axis



$\therefore$ focal distance $= SP = ePM$

$S'P = ePM'$

$\therefore$ Sum of focal distances $= SP + S'P = ePM + ePM'$

$= eNZ + eNZ'$

$= e(CZ - CN) + e(CN + CZ')$

$= e\left(\frac{a}{e} - x\right) + e\left(x + \frac{a}{e}\right) = 2a =$ constant

TEXT BOOK

1. Engineering Mathematics, H. K. Dass, S, Chand and Company Ltd, New Delhi, 2010

2. A Text Book of Comprehensive Mathematics Class XI, Parmanand Gupta, Laxmi Publications, New Delhi