Q.2 a. Show that if events A and B are independent, then
$$P(A \cap \overline{B}) = P(A)P(\overline{B})$$

Answer:

Events A and B are independent.

$$A = (A \cap B) \cup (A \cap \overline{B}) \quad \text{and} \quad (A \cap B) \cap (A \cap \overline{B}) = \varnothing$$

Therefore,

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

Since A and B are independent, we have

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A)P(\overline{B})$$

b. In a binary communication system (Fig.1), a 0 or 1 is transmitted. Because of channel noise, a 0 can be received as 1 and vice versa. Let m_0 and m_1 denote the events of transmitting 0 and 1, respectively. Let r_0 and r_1 denote the events of receiving 0 and 1, respectively. Let $P(m_0) = 0.5$, $P(r_1|m_0) = p = 0.1$, **and** $P(r_0|m_1) = q = 0.2$

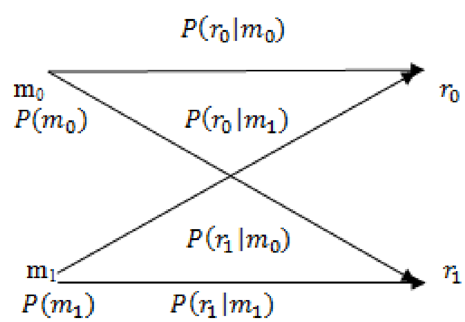


Fig. 1

(i) Find $P(r_0)$ and $P(r_1)$.
(ii) If a 0 was received, what is the probability that a 0 was sent?
(iii) If a 1 was received, what is the probability that a 1was sent?
(iv) Calculate the probability of error P_e.
(v) Calculate the probability that the transmitted signal is correctly read at the receiver.

Answer:

(i) From Fig. 1, we have

$$P(m_1) = 1 - P(m_0) = 1 - 0.5 = 0.5$$
$$P(r_0|m_0) = 1 - P(r_1|m_0) = 1 - p = 1 - 0.1 = 0.9$$
$$P(r_1|m_1) = 1 - P(r_0|m_1) = 1 - q = 1 - 0.2 = 0.8$$

Therefore,

$$P(r_0) = P(r_0|m_0)P(m_0) + P(r_0|m_1)P(m_1) = 0.9(0.5) + 0.2(0.5) = 0.55$$
$$P(r_1) = P(r_1|m_0)P(m_0) + P(r_1|m_1)P(m_1) = 0.1(0.5) + 0.8(0.5) = 0.45$$

(ii) Using Bayes' rule, we have

$$P(m_0|r_0) = \frac{P(r_0|m_0)P(m_0)}{P(r_0)} = \frac{(0.5)(0.9)}{0.55} = 0.818$$

(iii) Using Bayes' rule, we have

$$P(m_1|r_1) = \frac{P(r_1|m_1)P(m_1)}{P(r_1)} = \frac{(0.5)(0.8)}{0.45} = 0.889$$

(iv) The probability of error is given by

$$P_e = P(r_1|m_0)P(m_0) + P(r_0|m_1)P(m_1) = 0.1(0.5) + 0.2(0.5) = 0.15$$

(v) The probability that the transmitted signal is correctly read at the receiver is

$$P_c = P(r_0|m_0)P(m_0) + P(r_1|m_1)P(m_1) = 0.9(0.5) + 0.8(0.5) = 0.85$$

Q.3 a. Let X and Y be defined by

$X = \cos\theta$ **and** $Y = \sin\theta$

Where θ is a random variable uniformly distributed over [0,2π].

(i) Show that X and Y are uncorrelated

(ii) Show that X and Y are not independent

Answer:

(i) Given that

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Mean value of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} \cos\theta f_\Theta(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta d\theta = 0$$

Similarly

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{-\infty}^{\infty} \sin\theta f_\Theta(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta d\theta = 0$$

and

$$E[XY] = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta d\theta = 0 = E[X]E[Y]$$

Since $E[XY] = E[X]E[Y]$, X and Y are uncorrelated.

(ii)
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-\infty}^{\infty} \cos^2\theta f_\Theta(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}[1+\cos 2\theta]d\theta = \frac{1}{2}$$
$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_{-\infty}^{\infty} \sin^2\theta f_\Theta(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}[1-\cos 2\theta]d\theta = \frac{1}{2}$$
$$E[X^2Y^2] = \int_{-\infty}^{\infty} \cos^2\theta\sin^2\theta f_\Theta(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta = \frac{1}{8\pi}\int_0^{2\pi}\frac{1}{2}[1-\cos 4\theta]d\theta = \frac{1}{8}$$

Hence $E[X^2Y^2] \neq E[X^2]E[Y^2]$. If X and Y are independent, then $E[X^2Y^2] = E[X^2]E[Y^2]$. Therefore, X and Y are not independent.

b. Consider a random process Y(t) defined by

$$Y(t) = \int_0^t X(\tau)d\tau$$

Where X(t) is given by

$$X(t) = A\cos\omega t$$

Where ω is constant and $A = N[0;\sigma^2]$**.**

(i) Determine the pdf of Y(t) at t = t_k
(ii) Is Y(t) WSS?

Answer:

(i) $$Y(t_k) = \int_0^{t_k} X(\tau)d\tau = \int_0^{t_k} A\cos(\omega\tau)d\tau = A\frac{\sin\omega t_k}{\omega}$$

$Y(t_k)$ is a Gaussian random variable with

$$E[Y(t_k)] = E[A]\frac{\sin\omega t_k}{\omega} = 0$$
$$\sigma_Y^2 = var[Y(t_k)] = \sigma^2\left(\frac{\sin\omega t_k}{\omega}\right)^2$$

Hence pdf of $Y(t_k)$ is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{-y^2/2\sigma_Y^2}$$

(ii) Since the mean and variance of $Y(t)$ depend on time t_k, so $Y(t)$ is not WSS.

Q4 a. Explain about the average information content of symbols in long dependent sequences.

Answer: See Article 4.2.3 Page 145, of Text Book-I

b. Calculate the conditional entropy of an M-ary discrete memoryless channel.

Answer: Refer Pages 165-166 of Text Book-I

Q.5 a. A message source generates one of four messages randomly every microseconds. The probabilities of these messages are 0.4, 0.3, 0.2, and 0.1. Each emitted message is independent of the other messages in the sequence.
(i) What is the source entropy?
(ii) What is the rate of information generated by this source (in bits per second)?

Answer:

(i) $P_1 = 0.4$, $P_2 = 0.3$, $P_3 = 0.2$, and $P_4 = 0.1$.

$$H(m) = -(P_1 \log_2 P_1 + P_2 \log_2 P_2 + P_3 \log_2 P_3 + P_4 \log_2 P_4) = 1.846 \quad \text{bits/symbols}$$

The message source generates one of four messages randomly every microseconds, therefore, $r = 10^6$ symbiol/sec.

$$R = rH(X) = 1.846 \times 10^6 \quad \text{bits/sec}$$

b. An analog signal having 4-kHz bandwidth is sampled at 1.25 times the Nyquist rate, and each sample is quantized into one of 256 equally likely levels. Assume that the successive samples are statistically independent.
(i) What is the information rate of this source?
(ii) Can the output of this source be transmitted without error over an AWGN channel with a bandwidth of 10 kHz and S/N ratio of 20 dB?
(iii) Find the S/N ratio required for error free transmission for part (ii)

Answer:

(i)

$$f_m = 4 \times 10^3 \quad \text{Hz}$$
$$\text{Nyquist rate} = 2f_m = 8 \times 10^3 \quad \text{samples/sec}$$
$$r = 4 \times 10^3 \times 1.25 = 10^4 \quad \text{samples/sec}$$
$$H(X) = \log_2 256 = 8 \quad \text{b/sample}$$

The information of the source is given by

$$R = rH(X) = 8 \times 10^4 = 80 \quad \text{kb/sec}$$

(ii) The channel capacity is given by

$$C = B \log_2 \left(1 + \frac{S}{N}\right) = 10^4 \log_2(1 + 10^2) = 66.6 \times 10^3 \quad \text{b/sec}$$

Since $R > C$, error-free transmission is not possible.

(iii) The required S/N ratio can be found by

$$
\begin{aligned}
C = 10^4 \log_2 \left(1 + \frac{S}{N}\right) &\geq 8 \times 10^4 \\
\log_2 \left(1 + \frac{S}{N}\right) &\geq 8 \\
\left(1 + \frac{S}{N}\right) &\geq 2^8 \\
\frac{S}{N} &\geq 255 \quad (or \;\; 24.1 \; dB)
\end{aligned}
$$

Thus the required S/N ratio must be greater than or equal to 24.1 dB for error-free data transmission.

Q.6 a. Determine the capacity of a channel of infinite bandwidth.

Answer:

For white noise, the noise power $N = \eta B$. hence, as bandwidth B increases, N also increases. It can be shown that in the limit as $B \longrightarrow \infty$, C approaches a limit:

$$
\begin{aligned}
C &= B \log_2 \left(1 + \frac{S}{N}\right) \\
&= B \log_2 \left(1 + \frac{S}{\eta B}\right) \\
\lim_{B\to\infty} C &= \lim_{B\to\infty} B \log_2 \left(1 + \frac{S}{\eta B}\right) \\
&= \lim_{B\to\infty} \frac{S}{\eta} \left[\frac{\eta B}{S} \log_2 \left(1 + \frac{S}{\eta B}\right)\right]
\end{aligned}
$$

The limit can be found by noting that

$$\lim_{x\to\infty} \log_2 \left(1 + \frac{1}{x}\right) = \log_2 e = 1.44$$

Hence, $\lim_{B\to\infty} C = 1.44 \frac{S}{\eta}$ bits/sec

b. Consider a binary memoryless source X with two symbols x_1 and x_2. Show that H(X) is maximum when both x_1 and x_2 are equiprobable.

Answer:

Let $P(x_1) = \alpha$. then $P(x_2 = 1 - \alpha)$.

$$H(X) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$$
$$\frac{dH(X)}{d\alpha} = \frac{d}{d\alpha}\left[-\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)\right]$$

suing the relation

$$\frac{d}{dx}\log_b y = \frac{1}{y}\log_b e \frac{dy}{dx}$$

we obtain

$$\frac{dH(X)}{d\alpha} = -\log_2 \alpha + \log_2(1-\alpha) = \log_2 \frac{1-\alpha}{\alpha}$$

The maximum value of $H(X)$ requires that

$$\frac{dH(X)}{d\alpha} = 0$$
$$\text{that is} \qquad \frac{1-\alpha}{\alpha} = 1$$
$$\alpha = \frac{1}{2}$$

Thus, when $P(x_1) = P(x_2) = \frac{1}{2}$, $H(X)$ is maximum and is given by

$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 = 1 \quad \text{b/symbol}$$

Q.7 a. Consider an AWGN channel with 4-kHz bandwidth and the noise power spectral density $\frac{\eta}{2} = 10^{-12}$ W/Hz. The signal power required at the receiver is 0.1 mW. Calculate the capacity of this channel.

Answer:

$$B = 4000 \text{ Hz}, \qquad S = 0.1 \times 10^{-3} \text{ W}$$
$$N = \eta B = 2 \times 10^{-12} \times 4000 = 8 \times 10^{-9} \text{ W}$$

Thus

$$\frac{S}{N} = \frac{0.1 \times 10^{-3}}{8 \times 10^{-9}} = 1.25 \times 10^4$$

Channel capacity: $C = B\log_2\left(1 + \frac{S}{N}\right) = 4000\log_2(1 + 1.25 \times 10^4) = 54.44 \times 10^3$ b/sec

b. Draw and explain observations of Bandwidth-Efficiency diagram.

Answer: Refer Pages 48-49 of Text Book-II

Q.8 a. For a (6, 3) systematic linear block code, the three parity check bits c_4, c_5, and c_6 are formed from the following equation:

$$c_4 = d_1 \oplus d_3$$
$$c_5 = d_1 \oplus d_2 \oplus d_3$$
$$c_6 = d_1 \oplus d_2$$

(i) Write down the generator matrix G.
(ii) Construct all possible code words.
(iii) Suppose that the received word is 010111. Decode this received word by finding the location of the error and the transmitted data bits.

Answer:

(i) From the given equations, we obtain

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The generator matrix is given by

$$G = [I_3 \quad P] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(ii) Since $k = 3$, we have $2^3 = 8$ data words. Thus

$$c = dG = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

All possible code words are listed below.

d	c
000	000000
001	001110
010	010011
011	011101
100	100111
101	101001
110	110100
111	111010

(iii) From generator matrix, we get

$$H^T = \begin{bmatrix} P \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The received vector is $r = [0\ 1\ 0\ 1\ 1\ 1]$. The syndrome is given by

$$s = rH^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Since s is equal to the fourth row of H^T, an error is at the fourth bit, the correct code word is 010011, and the data bits are 010.

b. Given a generator matrix G = [1 1 1 1 1]. Construct a (5, 1) code. How many errors can this code correct? Find the codeword for data vectors d=0 and d=1. Comment on the result.

Answer:

We know that $c = dG$ where d is a single digit (0 or 1). For $d = 0$:

$$c = dG = 0\,[1\ \ 1\ \ 1\ \ 1\ \ 1] = [0\ \ 0\ \ 0\ \ 0\ \ 0]$$

For $d = 1$:

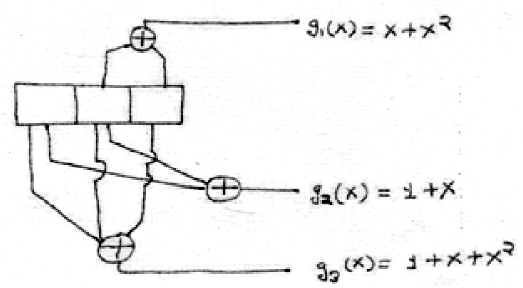
$$c = dG = 1\,[1\ \ 1\ \ 1\ \ 1\ \ 1] = [1\ \ 1\ \ 1\ \ 1\ \ 1]$$

Hence in this code a digit repeats 5 times. Such a code can correct up to two errors using majority rule for detection.

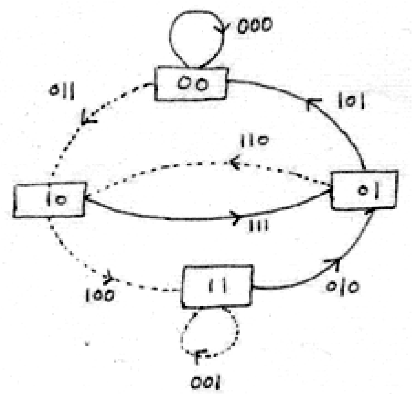
Q.9 a. Draw the state diagram, tree diagram, and trellis diagram for the K=3, rate 1/3 code generated by

$$\mathbf{g_1(X) = X + X^2}$$
$$\mathbf{g_2(X) = 1 + X}$$
$$\mathbf{g_3(X) = 1 + X + X^2}$$

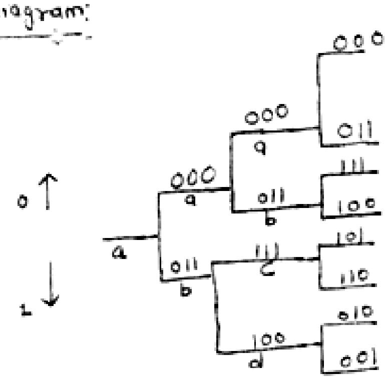
Answer:



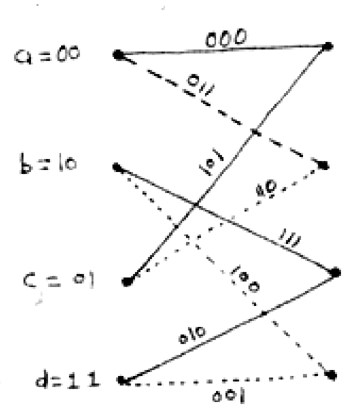
State diagram:



Tree diagram:



Trellis diagram:



b. Factorize the polynomial $x^3 + x^2 + x + 1$

Answer:

$$
\begin{array}{r|l}
 & x^2 + 1 \\
\hline
x+1 & x^3 + x^2 + x + 1 \\
 & x^3 + x^2 \\
\hline
 & \qquad\quad x + 1 \\
 & \qquad\quad x + 1 \\
\hline
 & \qquad\quad 0 \\
\hline
\end{array}
$$

Hence: $x^3+x^2+x+1 = (x+1)(x^2+1) = (x+1)(x+1)(x+1) = (x+1)^3$

TEXT BOOKS

I Digital and Analog Communication Systems by K. Sam Shanmugam, John Wiley India Edition, 2007 reprint.

II Digital Communications by Simon Haykin, John Wiley & Sons, Student Edition.