Q.2a. Show that the function \( f(z) = \sqrt{|xy|} \) is not analytic at the origin even though Cauchy-Riemann equations are satisfied thereof.

Soh: \( f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y) \), then

\[
\begin{align*}
& u(x, y) = \sqrt{|xy|}, \quad \text{and} \quad v(x, y) = 0 \\
& \frac{\partial u}{\partial x} = \frac{L}{x} \frac{u(x, 0) - u(0, 0)}{x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{L}{y} \frac{u(0, y) - u(0, 0)}{y}
\end{align*}
\]

At the origin, we have,

\[
\begin{align*}
& \frac{\partial u}{\partial x} = \frac{L}{x} \frac{0 - 0}{x} = 0 \quad \text{(1)} \\
& \frac{\partial u}{\partial y} = \frac{L}{y} \frac{0 - 0}{y} = 0 \quad \text{(1)}
\end{align*}
\]

\[
\frac{\partial v}{\partial x} = \frac{L}{x} \frac{u(0, y) - u(0, 0)}{x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{L}{y} \frac{u(0, y) - u(0, 0)}{y}
\]

\[
\begin{align*}
& \frac{\partial v}{\partial x} = \frac{L}{x} \frac{0 - 0}{x} = 0 \quad \text{(1)} \\
& \frac{\partial v}{\partial y} = \frac{L}{y} \frac{0 - 0}{y} = 0 \quad \text{(1)}
\end{align*}
\]

\[
\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}
\]

\[ \text{i.e. C.R. equations are satisfied at the origin.} \]

However \( f'(0) = \frac{L}{z} \left( \frac{f(z) - f(0)}{z-0} \right) \)

\[
\begin{align*}
& = \frac{L}{z} \sqrt{|xy|} \quad \text{when } z \to 0 \text{ along the line } y = mx \\
& = \sqrt{|m|} i \left( 1 + im \right) \quad \text{which is not unique.}
\end{align*}
\]

\[ \text{Hence } f(z) \text{ is not analytic at the origin.} \]
b. Find the bilinear transformation which maps the points $z_1 = 2, z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1, w_2 = i$ and $w_3 = -1$.

Q.3a. Find the Taylor’s series expansion of a function of the complex variable $f(z) = \frac{1}{(z-1)(z-3)}$ about the point, $z = 4$. Find its region of convergence.
We have, \( f(z) = \frac{1}{(z-1)(z-3)} \)

\[ f(4) = \frac{1}{3} \]
\[ f'(z) = \frac{1}{2} \left[ -\frac{1}{(z-3)^2} + \frac{1}{(z-1)^2} \right] \]
\[ f''(z) = \frac{1}{2} \left[ -\frac{2}{(z-3)^3} - \frac{2}{(z-1)^3} \right] \]

Putting \( z = 4 \),
\[ f''(4) = \frac{-4}{9} \]
\[ f'''(z) = \frac{1}{2} \left[ \frac{6}{(z-3)^4} - \frac{6}{(z-1)^4} \right] \]

Putting \( z = 4 \),
\[ f'''(4) = \frac{26}{27} \]

Similarly,
\[ f''''(4) = \frac{-80}{441} \]

We have Taylor's Series is,
\[ f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{1} f''(a) + \frac{(z-a)^3}{2} f'''(a) + \cdots \]

\[ \frac{1}{(z-1)(z-3)} = \frac{1}{3} - \frac{4}{9} (z-4) + \frac{13}{27} (z-4)^2 - \frac{340}{81} (z-4)^3 + \cdots \]
b. Determine the poles of the function \( f(z) = \frac{z^2}{(z-1)^2(z+2)} \) and find residue of \( f(z) \) of at each pole.
Q.4a. Evaluate \( \nabla e^z \) where \( r^2 = x^2 + y^2 + z^2 \).
b. Prove that \[ \text{div} \left( \frac{f(r) \cdot \vec{r}}{r} \right) = \frac{1}{r^2} \frac{d}{dr} \left( r^2 f \right) \]
Q.5  a. Apply Green’s theorem to evaluate \( \oint_C 2y^2\,dx + 3xdy \)

Where \( C \) is the boundary of closed region bounded between \( y = x \) and \( y = x^2 \)

b. Apply Stoke’s theorem to calculate \( \oint_C 4y\,dx + 2z\,dy + 6y\,dz \)

where \( C \) is the curve of intersection of \( x^2 + y^2 + z^2 = 6z \) and \( z = x + 3 \)
Q.6a. Using Lagrange’s interpolation formula, find the value of y corresponding to x = 10 from the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>5</th>
<th>6</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>16</td>
</tr>
</tbody>
</table>
b. Approximate the integral, \( I = \int_{0}^{1} e^x \, dx \) using Simpson’s \( \frac{1}{3} \)rd Rule with \( n = 8 \) interval.

Round off the result at 4 digits.
Q. 7a. Apply the Charpit method to solve the equation \( px + qy = pq \)
b. Using the method of separation of variable, solve \( \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \), where \( u(x,0) = 6e^{-3x} \)
\[
\frac{du}{dx} = 2 \frac{du}{dt} + u
\]

\[
\text{let } u = x(x) \cdot T(t)
\]

Where \( x \) is a function of \( x \) only and \( T \) is a function of \( t \) only.

Putting the value of \( u \) in \( i \), we get

\[
2 \left( x : T \right) \frac{dx}{dt} = 2 \frac{d}{dt} \left( x \cdot T \right) + x \cdot T
\]

\[
T \cdot \frac{dx}{dt} = 2x \frac{dT}{dt} + x \cdot T
\]

or

\[
T', x' = 2x \cdot T' + x \cdot T \text{ or } \frac{x'}{x} = 2 \frac{T'}{T} + 1 = \text{constant}
\]
Q.8  a. An urn ‘A’ contains 2 white and 4 black balls. Another urn ‘B’ contains 5 white and 7 black balls. A ball is transferred from the urn ‘A’ to the urn ‘B’, then a ball is drawn from urn ‘B’. Find the probability that it is white.
A uniform contains 2 white and 4 black balls.

A uniform B contains 5 white and 7 black balls.

Now there are two cases of transferring a ball from

A to B.

Case - I: when a white ball is transferred from A to B.

Probability (transfer of a white ball) = \( \frac{2}{2+4} = \frac{1}{3} \)

After transfer of a white ball, urn B contains 6 white balls and 7 black balls.

Probability (drawing a white ball from urn B after transfer) = \( P(\text{transfer of a white ball}) \times P(\text{drawing of a white ball}) \)

\[ \frac{1}{3} \times \frac{6}{13} = \frac{2}{13} \]

Case - II: when a black ball is transferred from A to B.

\( P(\text{transfer of a black ball}) = \frac{4}{2+4} = \frac{2}{3} \)

After transfer of a black ball, urn B contains 5 white and 8 black balls.

\( P(\text{drawing of a white ball from urn B after transfer}) = P(\text{transfer of black ball}) \times P(\text{drawing of a white ball}) \)

\[ = \frac{2}{3} \times \frac{5}{5+8} = \frac{10}{39} \]

\[ \text{Required probability} = \frac{2}{13} + \frac{10}{39} = \frac{16}{39} \text{ Ans.} \]
b. Assuming half the population of a town consumes chocolates and that 100 investigators each take 10 individuals to see whether they are consumers, how many investigators would you expect to report that three people or less were consumers?

Q.9 a. If the variance of the Poisson distribution is 2, find the probabilities for \( r = 1, 2, 3, 4 \) from the recurrence relation of the Poisson distribution. Also find \( P(r \geq 4) \).
b. The life of army shoes is ‘normally’ distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued how many pairs would be expected to need replacement after 12 months? [Given that $P(z \geq 2) = 0.0228$ and $z = \frac{x - \mu}{\sigma}$]
Text books