Q.2a. If $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that

$V_{xx} + V_{yy} + V_{zz} = m(m + 1)\, r^{m-2}$

Q.2.(a) $V = r^m = (x^2+y^2+z^2)^{\frac{m}{2}}$.

$$\therefore \frac{\partial V}{\partial x} = \frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1}.2x = mx(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$\frac{\partial^2 V}{\partial x^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-1} + mx\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}.2x$$

Similarly

$$\frac{\partial^2 V}{\partial y^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-1} + my\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}\,2y$$

$$\frac{\partial^2 V}{\partial z^2} = m(x^2+y^2+z^2)^{\frac{m}{2}-1} + mz\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}.2z$$

Adding all,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 3m(x^2+y^2+z^2)^{\frac{m-2}{2}} + m(m-2)(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{m}{2}-2}$$

$$= 3m\, r^{m-2} + m(m-2)\, r^{m-2} = m(m+1)\, r^{m-2}$$

b. If xyz = 8, find the values of x, y, z for which $u = \dfrac{5xyz}{x+2y+4z}$ is a maximum.

(b) If $xyz = 8$, $u = \dfrac{5xyz}{x+2y+4z} = \dfrac{40xy}{32+x^2y+2xy^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{40y}{32+x^2y+2xy^2} - \frac{40xy(2xy+2y^2)}{(32+x^2y+2xy^2)^2} = \frac{40y(32-x^2y)}{(32+x^2y+2xy^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{40x}{(32+x^2y+2xy^2)} - \frac{40xy(x^2+4xy)}{(32+x^2y+2xy^2)^2} = \frac{40x(32-2xy^2)}{(32+x^2y+2xy^2)^2}$$

Ext. value,

$\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 0$ which gives us $x = 2y$, $y = 2$, $x = 4$, $z =$

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{-80xy^2}{(32+x^2y+2xy^2)^2} + \frac{80y(32-x^2y)(2xy+2y^2)}{(32+x^2y+2xy^2)^3} = \frac{-80\times4\times4}{96\times96} = -\frac{5}{36} \text{ at } \begin{matrix} x=4 \\ y=2 \end{matrix}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{1280-80x^2y}{(32+x^2y+2xy^2)^2} - \frac{240y(32-x^2y)(x^2+4xy)}{(32+x^2y+2xy^2)^3} = -\frac{5}{36} \text{ at } x=4, y=2$$

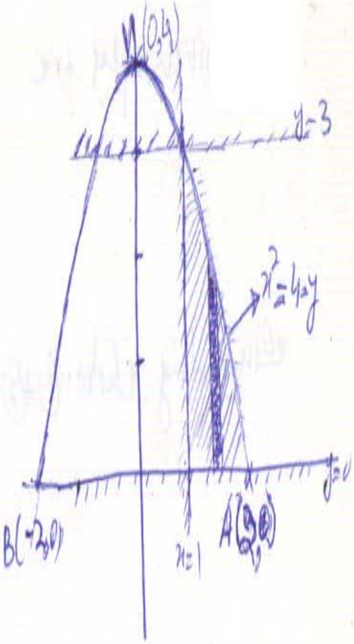
$$t = \frac{\partial^2 u}{\partial y^2} = \frac{-80x^2 2y}{(32+x^2y+2xy^2)^2} - \frac{80x(32-2xy^2)(x^2+4xy)}{(32+x^2y+2xy^2)^3} = -\frac{5}{9} \text{ at } x=4, y=2$$

since $rt - s^2 > 0$ and $r < 0$, so that at $x = 4$, $y = 2$, $z = 1$ is max

Q.3 a. Evaluate by changing order of integration of $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

-3-

Q.3 (b) Region of integration of given integral is $y=0$, $y=3$, $x=1$, $x=\sqrt{4-y}$ which is shown in the shaded figure. To change order of integration, we divide area by strips drawn parallel to y-axis.



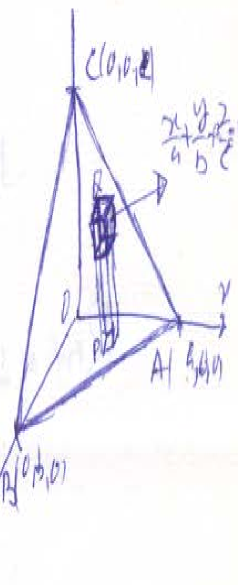
Now strip moves from $x=1$ to $x=2$, keeping its ends on $y=0$ and $y=4-x^2$. So changed order of integration is

$$\int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx = \int_1^2 \left| xy + \frac{y^2}{2} \right|_0^{4-x^2} dx = \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx$$

$$= \frac{241}{60}$$

b. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(b) OABC is the tetrahedron bounded by coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which meets the coordinates axes at $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$. Divide the volume into rectangular parallelopipeds of volume $\delta x \delta y \delta z$. To cover whole volume z moves from $z=0$ to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$, y from $y=0$, $y=b\left(1-\frac{x}{a}\right)$, $x=0$ to $x=a$. Hence volume is given by



$$\int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dx\,dy\,dz = \int_0^a\int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$= c\int_0^a\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{b^2}b^2\left(1-\frac{x}{a}\right)^2\right]dx = \frac{abc}{6}$$

MODERATION !

Q.4 a. Show that if $\lambda \neq -5$, the system of equations,

$$3x - y + 4z = 3,$$
$$x + 2y - 3z = -2,$$
$$6x + 5y + \lambda z = -3,$$

have a unique solution. If $\lambda = -5$, then show that the equations are consistent. Find the solutions in each case.

Q. 4 (a) The system of eqns can be rewritten in matrix form as

$R_3' = R_3 + 5R_1$, $R_2' = R_2 + 2R_1$ $\quad$ $R_3' = R_3 - 3R_2$

$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \text{ Or } \begin{bmatrix} 3 & -1 & 4 \\ 7 & 0 & 5 \\ 21 & 0 & 20+\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix} \text{ OR } \begin{bmatrix} 3 & -1 & 4 \\ 7 & 0 & 5 \\ 0 & 0 & 5+\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

~~It~~ clearly these eqns have unique sol only $5+\lambda \neq 0$ (i.e. Rank of Coeff. matrix is 3) i.e if $\lambda \neq -5$ and in this case $z = 0$, $x = \frac{4}{7}$ and $y = -\frac{9}{7}$.

If $5+\lambda = 0$, rank of coeff matrix = Rank of augmented matrix = 2, ∴ eqns are consistent and have infinite solutions. Solutions are given by $7x + 5z = 4$ and $3x - y + 4z = 3$

Taking $z = k$, $x = \frac{4-5k}{7}$, $y = (13k-9)/7$

b. Use Gauss elimination method to solve the equations,

$$x - y + z = 6$$
$$2x + 4y + z = 3$$
$$3x + 2y - 2z = -2$$

(b) Given eqns are $x - y + z = 6$ — (1)

$2x + 4y + z = 3$ — (2)

$3x + 2y - 2z = -2$ — (3)

eliminating x from: (1)+(2) and (1)+(3),

$6y - z = -9$ — (4)

$5y - 5z = -20$ — (5)

eliminating y from (4)+(5), $-25z = -75$

By Back Substitution,

$z = 3,\ y = -1,\ x = 2$

Q.5 a. Develop Newton – Raphson formula for finding $\sqrt{N}$, where N is a real number. Use it to find $\sqrt{41}$. Correct to 3 decimal places.

Q.5 (a) Let $x = \sqrt{N}$, $\therefore x^2 = N$ or $f(x) = x^2 - N = 0$

By Newton-Raphson method, $(n+1)^{th}$ iteration of $f(x)=0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

Hence

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \quad \text{—(1)}$$

Root of 41 lies between 6 and 7. Let 1st approx. root = 6.5

$\therefore$ by (1),

$$x_2 = \frac{1}{2}\left[6.5 + \frac{41}{6.5}\right] = 6.40385$$

$$x_3 = \frac{1}{2}\left[6.40385 + \frac{41}{6.40385}\right] = 6.403124$$

It shows that value of $\sqrt{41}$ correct to 3 decimal places is 6.403.

MODERATION-T

b. Use Runge – Kutta method of order four for find y at x = 0.2 given that

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, \; y(0) = 1$$

taking h = 0.2

(b) Here $f(x,y) = \dfrac{y^2-x^2}{y^2+x^2}$, $x_0 = 0, y_0 = 1, h = 0.2$

$\Rightarrow K_1 = hf(x_0, y_0) = (0.2)\dfrac{1^2-0^2}{1^2+0^2} = 0.2$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = (0.2)\dfrac{(1.1)^2-(0.1)^2}{(1.1)^2+(0.1)^2} = 0.1967$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = .2 \times \dfrac{(1.0983)^2-(.1)^2}{(1.0983)^2+(.1)^2} = 0.1967$

$K_4 = hf(x_0+h, y_0+K_3) = .2 \times \dfrac{(1.1967)^2-(.2)^2}{(1.1967)^2+(.2)^2} = 0.1891$

$K = \dfrac{K_1+2K_2+2K_3+K_4}{6} = 0.196$

Hence $y(0.2) = y_0 + K = 1.196$

Q.6 a. Solve the differential equation $(1+x+y)^2 \dfrac{dy}{dx} = 1$

6(a) Put $1+x+y = t$, $\therefore 1+\dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$

Substituting it in given eqn.

$t^2\left(\dfrac{dt}{dx}-1\right) = 1$ or $t^2\dfrac{dt}{dx} = 1+t^2$

or $\dfrac{t^2}{1+t^2}\,dt = dx$ or $\dfrac{t^2+1-1}{1+t^2}\,dt = dx$

Integrating $t - \tan^{-1} t = x + c$

Hence $1+x+y - \tan^{-1}(1+x+y) = x + c$

or $1+y = \tan^{-1}(1+x+y) + c$

b. Show that the family of parabolas $x^2 = ua\ (y + a)$ is self orthogonal

(b) Since $x^2 = 4a(y+a)$, $\therefore\ 2x = 4a\frac{dy}{dx}$

eliminating 'a', we get

$$x^2 = \frac{2x}{\frac{dy}{dx}}\left(y + \frac{x}{2\frac{dy}{dx}}\right)$$

Or

$$x\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx} + x \quad \text{—(1)}$$

(1) is diff. eqn. of given system. To diff. eqn. of orthogonal system, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (1), so diff. eqn. of orthogonal system is

$$x\left(-\frac{dx}{dy}\right)^2 = 2y\left(-\frac{dx}{dy}\right) + x$$

Or

$$x = -2y\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2$$

Or

$$x\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx} + x \quad \text{—(2)}$$

(2) is same as (1). Hence orthogonal system is same as given system. Hence given system is self orthogonal

Q.7 a. Solve the equation $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$

7(a) A.E. is $m^3 + 2m^2 + m = 0$ and its roots are $0, -1, -1$

$\therefore CF = c_1 + (c_2 + c_3 x)e^{-x}$.

and $P.I = \frac{1}{D^3+2D^2+D}\cdot\left(x^2 e^{2x} + \sin^2 x\right)$

$= e^{2x}\frac{1}{(D+2)^3+2(D+2)^2+D+2}\cdot x^2 + \frac{1}{2}\frac{1}{D^3+2D^2+D}(1-\cos 2x)$

$= e^{2x}\frac{1}{D^3+8D^2+21D+18}\cdot x^2 + \frac{x}{2} - \frac{1}{2(-4D-8+D)}\cos 2x$

$= \frac{e^{2x}}{18}\left[1+\frac{D^3+8D^2+21D}{18}\right]^{-1} x^2 + \frac{x}{2} + \frac{1}{2(3D+8)}\cos 2x$

$= \frac{e^{2x}}{18}\left[1 - \frac{D^3+8D^2+21D}{18} + \left(\frac{D^3+8D^2+21D}{18}\right)^2\right]x^2 + \frac{x}{2} + \frac{1}{2}\frac{(3D-8)}{-36-64}\cos 2x$

$= \frac{e^{2x}}{18}\left[x^2 - \frac{8.2}{18} - \frac{21.2x}{18} + \frac{21.21.2}{18.18}\right] + \frac{x}{2} - \frac{1}{200}\left(-6\sin 2x - 8\cos 2x\right)$

$= \frac{e^{2x}}{18}\left[x^2 - \frac{7}{3}x + \frac{11}{6}\right] + \frac{x}{2} + \frac{1}{100}\left(3\sin 2x + 4\cos 2x\right)$

Hence complete soln

$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7}{3}x + \frac{11}{6}\right) + \frac{x}{2} + \frac{1}{100}\left(3\sin 2x + 4\cos 2x\right)$

b. Use method of variation of parameters to solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$

7(b) A.E. of given eqn. is $m^2-2m+1=0$ i.e. $m=1,1$

∴ C.F. $=(c_1+c_2x)e^x = c_1e^x + c_2xe^x$

To find P.I. by method of variation of parameters, let

$y_1 = e^x$, $y_2 = xe^x$, RHS $= X = \frac{e^x}{x}$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix} = e^{2x}$

$\therefore P.I. = -y_1\int \frac{y_2X}{W}dx + y_2\int \frac{y_1X}{W}dx = -e^x\int \frac{xe^x}{e^{2x}}\cdot\frac{e^x}{x}dx + xe^x\int\frac{e^x}{e^{2x}}\cdot\frac{e^x}{x}dx$

$= -xe^x + xe^x \log x.$

∴ C.S. is $y = (c_1+c_2x)e^x - xe^x + xe^x\log x = (c_1 + c_2'x)e^x + xe^x\log x$

Q.8 a. Find the series solution of the differential equation $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0$

Q8(a) Here $x=0$ is a singular point, so let the solution be

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$

$\therefore \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots$

$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots$

Substituting these in given diff eqn, we get

$$2x^2\left[m(m-1)a_0x^{m-2} + (m+1)m\,a_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots\right]$$
$$-x\left[m a_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots\right] + (1-x^2)\left(a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots\right) = 0$$

Indicial eqn is obtained by equating to zero the coeffts of lowest power of x, we get

$$2m(m-1)a_0 - m a_0 + a_0 = 0 \quad \text{i.e. } m = 1, \tfrac{1}{2} \quad (\because a_0 \neq 0)$$

equating to zero the coeffts of successive powers of x, we get

$$2(m+1)m\,a_1 - (m+1)a_1 + a_1 = 0 \quad \therefore a_1 = 0 \quad (\because m = 1, \tfrac{1}{2})$$

$$2(m+2)(m+1)a_2 - (m+2)a_2 + a_2 - a_0 = 0 \quad \text{or } a_2 = \frac{a_0}{(m+2)(2m+1)+1} = \frac{a_0}{(m+1)(2m+3)}$$

$$2(m+3)(m+2)a_3 - (m+3)a_3 + a_3 - a_1 = 0 \quad \text{or } a_3 = \frac{a_1}{(m+3)(2m+3)+1} = \frac{a_1}{(m+2)(2m+5)} = 0$$

$$2(m+4)(m+3)a_4 - (m+4)a_4 + a_4 - a_2 = 0 \quad \text{or } a_4 = \frac{a_2}{(m+3)(2m+7)} = \frac{a_0}{(m+1)(m+3)(2m+3)(2m+7)}$$

When $m = 1$, $a_2 = \frac{a_0}{2.5}$, $a_4 = \frac{a_0}{2.4.5.9}$, $a_6 = \frac{a_0}{2.4.6.5.9.13}$ etc

When $m = \frac{1}{2}$, $a_2 = \frac{a_0}{2.3}$, $a_4 = \frac{a_0}{2.4.3.7}$, $a_6 = \frac{a_0}{2.4.6.3.7.11}$ etc

$\therefore$ If $m=1$, soln is

$$y_1 = a_0x + \frac{a_0}{2.5}x^3 + \frac{a_0}{2.4.5.9}x^5 + \cdots = a_0x\left(1 + \frac{x^2}{2\times5} + \frac{x^4}{2\times4\times5\times9} + \cdots\right)$$

If $m = \frac{1}{2}$, soln is

$$y_2 = a_0x^{1/2} + \frac{a_0}{2.3}x^{5/2} + \frac{a_0}{2.4.3.7}x^{9/2} + \cdots = a_0\sqrt{x}\left(1 + \frac{x^2}{2.3} + \frac{x^4}{2.4.3.7} + \cdots\right)$$

Hence complete soln is

$$y = c_1y_1 + c_2y_2 = c_1x\left(1 + \frac{x^2}{2\times5} + \frac{x^4}{2\times4\times5\times9} + \cdots\right)$$
$$+ c_2\sqrt{x}\left(1 + \frac{x^2}{2\times3} + \frac{x^4}{2\times4\times3\times7} + \cdots\right)$$

b. Show that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \beta(m+1, n+1)$

(b) To evaluate $\int_a^b (x-a)^m (b-x)^n dx$, put $x-a=t$, $\therefore dx = dt$

$$= \int_0^{b-a} t^m (b-a-t)^n dt, \text{ now put } t=(b-a)z$$

$$= (b-a)^m \int_0^1 z^m (b-a)^n (1-z)^n (b-a) dz$$

$$= (b-a)^{m+n+1} \int_0^1 z^{m+1-1} (1-z)^{n+1-1} dz = (b-a)^{m+n+1} \beta(m+1, n+1)$$

Q.9 a. Prove that $P_n(x) = \frac{1}{n} \frac{1}{2^n} \frac{d^n}{dx^n} \left(x^2-1\right)^n$

Q 9(a) Let $v = (x^2-1)^n$

$\therefore v_1 = \frac{dv}{dx} = n(x^2-1)^{n-1} 2x = \frac{2nx(x^2-1)^n}{x^2-1}$

or $(x^2-1)v_1 - 2nx(x^2-1)^n = 0$ or $(1-x^2)v_1 + 2nxv = 0$

Diff. it $(n+1)$ times by Leibnitz's method, we get

$$(1-x^2)v_{n+2} + (n+1)v_{n+1}(-2x) + \frac{(n+1)n}{2} v_n(-2) + 2nxv_{n+1} + 2n(n+1)v_n = 0$$

or $(1-x^2)\frac{d^2 v_n}{dx^2} - 2x\frac{d v_n}{dx} + n(n+1)v_n = 0$

It is Legendre's equation of order n and Cv_n is its solution. Also $P_n(x)$ is its solution.

$$\therefore P_n(x) = Cv_n = C\frac{d^n}{dx^n}(x^2-1)^n$$

To find C, put $x=1$,

$$\therefore P_n(1) = 1 = C\left[\frac{d^n (x^2-1)^n}{dx^n}\right]_{at\, x=1}$$

$$= C\left[\frac{d^n}{dx^n}(x-1)^n(x+1)^n\right]_{x=1}$$

$$= C\left[\underline{|n}\,(x+1)^n + \text{terms containing } (x-1) \text{ and its powers}\right]_{x=1}$$

$$= C\,\underline{|n}\,2^n, \quad \therefore C = \frac{1}{2^n \underline{|n}}$$

Hence

$$P_n(x) = \frac{1}{2^n \underline{|n}} v_n = \frac{1}{2^n \underline{|n}} \frac{d^n}{dx^n}(x^2-1)^n$$

b. Show that :

$$\frac{d}{dx}\left\{J_n^2(x) + J_{n+1}{}^2(x)\right\} = 2\left\{\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J_{n+1}^2(x)\right\}$$

(b) $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_n(x)J_n'(x) + 2J_{n+1}(x)J_{n+1}'(x)$

$= 2J_n(x)\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}(x)\left[J_n - \frac{n+1}{x}J_{n+1}(x)\right]$

$= 2J_n^2(x)\frac{n}{x} - 2J_{n+1}^2\frac{n+1}{x}$

$= 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J_{n+1}^2\right]$

using $\frac{d}{dx}(x^n J_n) = x^n J_{n-1}(x)$

$\therefore J_n'(x) = \frac{x^n J_{n-1}(x) - n x^{n-1} J_n(x)}{x^n}$

+ putting $n = n+1$

and using $\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}(x)$

$\therefore J_n' = \frac{-n x^{-n-1} J_n - x^{-n} J_{n+1}}{x^{-n}}$

$= \frac{n}{x}J_n - J_{n+1}$

Textbooks

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi

2. Text book of Engineering Mathematics, NP Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd.