Q.2 a. Show that the real and imaginary parts of the function $w = \log z$ satisfy the Cauchy – Riemann equations when z is not zero. Find its derivatives. **(8)**

Answer:

Here $w = \log z$
$= \log(x+iy)$

To separate the real and imaginary parts, put $x = r\cos\theta$, $y = r\sin\theta$ and $w = u+iv$ i.e.

$$w = \log z \Rightarrow u+iv = \log(r\cos\theta + ir\sin\theta)$$
$$= \log_e re^{i\theta}$$
$$= \log_e r + \log_e e^{i\theta}$$
$$= \log_e r + i\theta$$
$$= \log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$\left[\because r^2 = x^2+y^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right.$

or
$$u+iv = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow u = \frac{1}{2}\log_e(x^2+y^2);\quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Now
$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{(x^2+y^2)}\cdot(2x) = \frac{x}{x^2+y^2} \quad \text{— (i)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \quad \text{—— (ii)}$$

From (i) and (ii) $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad$ —— (A)

Again
$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \text{ — (iii)},\quad \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} \text{ — (iv)}$$

From (iii) & (iv)
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \text{—— (B)}.$$

Hence From A and B are C–R equation and P. deff. are Continuous

Hence $w = \log z$ is analytic function except (7)

when $x^2+y^2 = 0 \Rightarrow x=0=y=0 \Rightarrow x+iy=0$

Now $w = u+iv$ $\Rightarrow z = 0$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z}$$

Hence $\frac{dw}{dz} = \frac{1}{z}$, which is the required derivatives.

b. Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along the path (i). $y = x$ (ii). $y = x^2$. **(8)**

Answer:

Let $I = \int_0^{1+i} (x^2 - iy)\, dz$

$$= \int (x^2 - iy)(dx + i\,dy)$$

(i) along the path $y = x$, then $dy = dx$ and $x = 0$ to 1 then

$$I_1 = \int_0^1 (x^2 - ix)(dx + i\,dx)$$

$$= \int_0^1 (x^2 - ix)(1+i)\, dx$$

$$= (1+i)\left(\frac{x^3}{3} - i\frac{x^2}{2}\right)_0^1$$

$$= (1+i)\left(\frac{1}{3} - \frac{i}{2}\right) = \frac{5}{6} - \frac{i}{6}$$

and

(ii) Along the path $y = x^2$ then $dy = 2x\,dx$ and $x = 0$ to 1. then

$$I_2 = \int_0^1 (x^2 - ix^2)(dx + i\,2x\,dx)$$
$$= \int_0^1 x^2(1-i)(1+2ix)\,dx$$
$$= (1-i)\int_0^1 x^2(1+2ix)\,dx$$
$$= (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$
$$= (1-i)\left(\frac{1}{3} + \frac{i}{2}\right)$$
$$= \frac{5}{6} + \frac{i}{6}$$

Q.3 a. Find the bilinear transformation, which maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ in to $w_1 = i, w_2 = -1, w_3 = -i$ **(8)**

Answer:

Here $z_1 = 0, z_2 = 1, z_3 = \infty$ and $w_1 = i$, $w_2 = -1$, $w_3 = -i$.

Here $z_3 = \infty$, so that we can not apply the formula

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \text{ directly}$$

so we can write it as

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{(z - z_1)\left(\frac{z_2}{z_3} - 1\right)}{\left(\frac{z}{z_3} - 1\right)(z_2 - z_1)}$$

Now substituting the all values

$$\frac{w - i}{w + i} \cdot \frac{-1 + i}{-1 - i} = \frac{(z - 0)(0 - 1)}{(0 - 1)(1 - 0)} \Rightarrow \frac{w - i}{w + i}(-i) = z$$

solving for w, then

$$w = -i\,\frac{z - i}{z + i} = -\left(\frac{iz + 1}{z + i}\right) \quad \text{Ans.}$$

b. Find the terms in the Laurent expansion of $f(z) = \dfrac{1}{(z+1)(z+3)}$, for the region (i) $1 < |z| > 3$ (ii) $|z| < 1$ **(8)**

Answer:

Solution (b). Here $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

Case (i) $1 < |z| < 3 \Rightarrow \frac{1}{z} < 1$ and $\frac{z}{3} < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{z(1+\frac{1}{z})} - \frac{1}{3(1+\frac{z}{3})}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots\right) - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

$$= \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

which is the required series when $1 < z < 3$

Case (ii). $|z| < 1$, then

$$f(z) = \frac{1}{2(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{2}(3+z)^{-1}$$

$$= \frac{1}{2}(1-z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} \quad \left[\because |z|<1,\ \frac{z}{3}<1\right.$$

$$f(z) = \frac{1}{2}(1-z+z^2-z^3+ \cdots) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

$$= \left(\frac{1}{2}-\frac{1}{6}\right) + \left(-\frac{z}{2}+\frac{z}{18}\right) + \left(\frac{z^2}{2}-\frac{z^2}{54}\right) + \left(-\frac{z^3}{2}+\frac{z^3}{162}\right) + \cdots$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

which is the required expansion.

Q.4 a. If $\vec{r} = xi + yj + zk$, a = $|\vec{r}|$ and $\vec{a}$ is a constant vector, then find the value of $div\left[\frac{\vec{a} \times \vec{r}}{r^n}\right]$ **(8)**

Answer:

Here $\vec{r} = xi + yj + zk$, then

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ and

Let $\vec{a} = a_1 i + a_2 j + a_3 k$

Now
$$\vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i(a_2 z - a_3 y) - j(a_1 z - a_3 x) + k(a_1 y - a_2 x)$$

and $$\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n} = \frac{(a_2z-a_3y)i-(a_1z-a_3x)j+(a_1y-a_2x)k}{(x^2+y^2+z^2)^{\frac{n}{2}}}$$

$$\text{div}\left[\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}\right] = \nabla\cdot\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\frac{(a_2z-a_3y)i-(a_1z-a_3x)j+(a_1y-a_2x)k}{(x^2+y^2+z^2)^{n/2}}$$

$$= \frac{\partial}{\partial x}\frac{(a_2z-a_3y)}{(x^2+y^2+z^2)^{n/2}} - \frac{\partial}{\partial y}\frac{(a_1z-a_3x)}{(x^2+y^2+z^2)^{n/2}} + \frac{\partial}{\partial z}\frac{(a_1y-a_2x)}{(x^2+y^2+z^2)^{n/2}}$$

$$= -\frac{n}{2}\frac{(a_2z-a_3y)2x}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} + \frac{n}{2}\frac{(a_1z-a_3x)2y}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} + \frac{n}{2}\frac{(a_1y-a_2x)2z}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}$$

$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}\left[(a_2z-a_3y)x-(a_1z-a_3x)y+(a_1y-a_2x)z\right]$$

$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}\left[a_2zx-a_3xy-a_1yz+a_3xy+a_1yz-a_2xz\right]$$

$$= 0$$

Hence $\text{div}\left[\frac{\vec{a}\times\vec{r}}{r^n}\right] = 0$

b Define CURL of a vector point function with physical interpretation. **(8)**

Answer:

Definition of CURL:

Let $\vec{F} = \vec{F_1} i + \vec{F_2} j + \vec{F_3} k$ be any vector point function $\vec{F}$, then the curl of a vector point function $\vec{F}$ is defined as below i.e.

$$\text{curl } \vec{F} \text{ or } \nabla \times \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(\vec{F_1} i + \vec{F_2} j + \vec{F_3} k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - j\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + k\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

curl $\vec{F}$ is a vector quantity.

Physical Interpretation of curl:-

Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ is the angular velocity and $\vec{r}$ is the position vector of a point on a rotating body. i.e., $\vec{r} = xi + yj + zk$

We know that $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{V}$ is the linear velocity, then

$$\text{curl } \vec{V} = \nabla \times \vec{V} \Rightarrow \nabla \times (\vec{\omega} \times \vec{r})$$

$$= \nabla \times [(w_1 i + w_2 j + w_3 k) \times (x i + y j + z k)]$$

(207/2) (13)

$$= \nabla \times \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(w_2 z - w_3 y) i - (w_1 z - w_3 x) j + (w_1 y - w_2 x) k]$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) \times [(w_2 z - w_3 y) i - (w_1 z - w_3 x) j + (w_1 y - w_2 x) k)]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (w_2 z - w_3 y) & (w_1 z - w_3 x) & (w_1 y - w_2 x) \end{vmatrix}$$

$$= (w_1 + w_1) i - (-w_2 - w_2) j + (w_3 + w_3) k$$

$$= 2(w_1 i + w_2 j + w_3 k)$$

$$= 2w$$

$\therefore$ curl $\vec{v} = 2w$, which shows that curl of a vector field is connected with rotational properties of the vector field.

If curl $\vec{F} = 0$, the field $\vec{F}$ is termed as irrotational.

Q.5 a. Evaluate $\int_c \vec{F} \bullet d\vec{r}$, where $\vec{F} = \dfrac{iy - jx}{x^2 + y^2}$ and c is the circle $x^2 + y^2 = 1$ traversed counter clockwise. **(8)**

Answer:

Let $\vec{r} = xi + yj + zk$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

Here $\vec{F} = \dfrac{iy - jx}{x^2+y^2}$

Now $\vec{F}.d\vec{r} = \left(\dfrac{iy - xj}{x^2+y^2}\right).(dx\,i + dy\,j + dz\,k)$

$= \dfrac{dx\,y}{x^2+y^2} - \dfrac{x\,dy}{x^2+y^2} \Rightarrow \dfrac{y\,dx - x\,dy}{x^2+y^2}$

$\int_c \vec{F}.d\vec{r} = \int_c y\,dx - x\,dy \quad \because x^2+y^2 = 1$

Now parametric equation of the circle

$x = r\cos\theta,\ y = r\sin\theta \ ; r = 1$

$dx = -r\sin\theta\,d\theta,\ dy = r\cos\theta\,d\theta$

and θ move from 0 to 2π

$\therefore \int_c \vec{F}.d\vec{r} = \int_0^{2\pi} r\sin\theta(-r\sin\theta\,d\theta) - r\cos\theta(r\cos\theta\,d\theta)$

Put $r = 1$

$= \int_0^{2\pi} (-\sin^2\theta - \cos^2\theta).d\theta$

$= \int_0^{2\pi} -d\theta = \left[-\theta\right]_0^{2\pi} = \underline{\underline{-2\pi}}$

Hence $\int_c \vec{F}.d\vec{r} = \underline{\underline{-2\pi}}$

b. Verify Stocke's Theorem for $\vec{F} = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the x-y plane. **(8)**

Answer:

we know that Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds$$

Now $\int_C \vec{F}\cdot d\vec{r}$, where C is the boundary of hemisphere i.e. circle the ~~circle~~ $x^2+y^2+z^2=16$

and $\vec{F} = (x^2+y-4)i + 3xyj + (2xz+z^2)k$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x^2+y-4)\,i + 3xy\,j + (2xz+z^2)\,k\right]\cdot(dx\,i + dy\,j)$$

$$= \int_C \left[(x^2+y-4)\,dx + 3xy\,dy\right]$$

Now parametric equation of circle

$x = 4\cos\theta \Rightarrow dx = -4\sin\theta\, d\theta$

$y = 4\sin\theta \Rightarrow dy = 4\cos\theta\, d\theta$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left[(16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\, d\theta) + 3(4\cos\theta)(4\sin\theta)(4\cos\theta)\, d\theta\right]$$

$$= 16\int_0^{2\pi} (-4\cos^2\theta\sin\theta - \sin^2\theta + \sin\theta + 12\sin\theta\cos^2\theta)\, d\theta$$

$$= 16\int_0^{2\pi} (8\sin\theta\cos^2\theta - \sin^2\theta + \sin\theta)\, d\theta$$

$$= -16\int_0^{2\pi} \sin^2\theta\, d\theta \qquad \left(\because \int_0^{2\pi} \sin^n\theta\cos\theta\, d\theta = 0,\ \int_0^{2\pi} \cos^n\theta\sin\theta\, d\theta = 0\right)$$

MODERATION-I

$$= -16 \times 4 \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= -64 \left(\frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$\int_C \vec{F} \cdot d\vec{r} = \underline{\underline{-16\pi}}$$

Now $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$

Now $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z \end{vmatrix}$

$$= i(0-0) - j(2z-0) + k(3y-1)$$

$$\nabla \times \vec{F} = -2zj + (3y-1)k$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-16)}{|\nabla \phi|}$$

$$= \frac{2xi + 2yj + 2zk}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}} = \frac{xi + yj + kz}{4}$$

$$\therefore (\nabla \times \vec{F}) \cdot \hat{n} = (-2zj + (3y-1)k) \cdot \left(\frac{xi + yj + zk}{4}\right)$$

$$= \underline{\frac{-2yz + (3y-1)z}{4}}$$

$$\hat{k} \cdot \hat{n} \, ds = dx\,dy \Rightarrow \frac{xi + yj + zk}{4} \cdot k \, ds = dx\,dy$$

$$\Rightarrow \frac{z}{4} ds = dx\,dy$$

$$= ds = \frac{dx\,dy}{} \times \frac{4}{z}$$

Hence $\iint (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint \frac{-2yz + (3y-1)z}{4} \times \frac{4}{z}\, dx\,dy$

$$= \iint [-2y + (3y-1)]\, dx\,dy$$

$$= \iint (y-1)\, dx\,dy$$

on putting, $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta$

$$= \iint (r\sin\theta - 1)\, r\, d\theta\, dr$$

$$= \int d\theta \int (r^2\sin\theta - r)\, dr$$

$$= \int_0^{2\pi} d\theta \left(\frac{r^3}{3}\sin\theta - \frac{r^2}{2}\right)_0^4$$

$$= \int_0^{2\pi} d\theta \left(\frac{16}{3}\sin\theta - 8\right)$$

$$= \left[-\frac{64}{3}\cos\theta - 8\theta\right]_0^{2\pi}$$

$$= -\frac{64}{3} - 16\pi + \frac{64}{3}$$

$$= \underline{-16\pi}$$

Hence the line integral is equal to the surface integral, hence stoke's theorem is verified.

Q.6 a. State and Prove Lagrange's interpolation formula. **(8)**

Answer:

Statement:- Let $y = f(x)$ be a function, which takes the values $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ corresponding to the values of $x = x_0, x_1, x_2, \ldots x_n$, not necessarily equal spaced, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Proof: Let $y = f(x)$ be a function which can assume the value $f(x_0), f(x_1) \ldots f(x_n)$ corresponding to the values of the argument $x_0, x_1, x_2 \ldots x_n$ respectively.

Since there are $(n+1)$ pairs of values of x and y, therefore, we can represent $f(x)$ by a polynomial in x of degree n.

ie.

$$f(x) = A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_2)\cdots(x-x_n) + A_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + \cdots + A_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \quad \text{(A)}$$

where $A_0, A_1, A_2, A_3 \ldots A_n$ are constants

It can be determined by putting $f(x) = f(x_0)$ at $x = x_0$, $f(x) = f(x_1)$ at $x = x_1$ and ...

Put $x = x_0$ and $f(x) = f(x_0)$ in equation(A) (19)

$$f(x_0) = A_0(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

$$\therefore A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

again put $x = x_1$ and $f(x) = f(x_1)$ in (A), then

$$f(x_1) = A_1(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)$$

$$\therefore A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

Proceeding in the same way, we get the value of $A_1, A_2, A_3 \cdots A_n$ i.e.

$$A_i = \frac{f(x_i)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_n)}$$

Putting all values of $A_0, A_1, A_2 \cdots A_i \cdots A_n$ in equation (A), we get

$$f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} f(x_1) + \cdots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} f(x_n)$$

which is the required Lagrange's interpolation formula.

b. Evaluate $\int_{0.5}^{0.7} \sqrt{x}\, e^{-x}\, dx$ by Simpson's $\frac{1}{3}$ rule. **(8)**

Answer:

Let $I = \int_{0.5}^{0.7} \sqrt{x}\, e^{-x}\, dx$

Here interval is $(0.5, 0.7)$, on dividing the interval in to 4 equal parts each of width $h = \frac{0.7 - 0.5}{4} = 0.05$

Here $f(x) = \sqrt{x}\, e^{-x}$, Now

x	x_0 0.5	x_1 0.55	x_2 0.60	x_3 0.65	x_4 0.7
$y = \sqrt{x}\, e^{-x}$	0.42888 y_0	0.42787 y_1	0.42511 y_2	0.42088 y_3	0.41547 y_4

Let $x = x_0, x_1, x_2, x_3, x_4$ are 0.5, 0.55, 0.60, 0.65 and 0.70, then the value of y are y_0, y_1, y_2, y_3, y_4 i.e. 0.42888, 0.42787, 0.42511, 0.42088 and 0.41547.

Now Simpson's $\frac{1}{3}$ rule is

$$\int_{0.5}^{0.7} \sqrt{x}\, e^{-x}\, dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$= \frac{0.05}{3}\left[0.42888 + 4(0.42787 + 0.42088) + 2(0.42511) + 0.41547\right]$$

$$= \frac{0.05}{3}\left[0.84435 + 4(0.84875) + 0.85022\right]$$

$$= \frac{0.05}{3}(5.08957)$$

$$= 0.08483$$

Q.7 a. Form the partial differential equation by eliminating the function f from the relation $z = y^2 + 2\ f(\frac{1}{x} + \log y)$. **(8)**

Answer:

Here $z = y^2 + 2f(\frac{1}{x} + \log y)$ —— ①

For formation the P.D.E,

Differentiate ① w.r.to x & y as partially

$$p = \frac{\partial z}{\partial x} = 2f'(\frac{1}{x} + \log y) \cdot (-\frac{1}{x^2})$$

$$-px^2 = 2f'(\frac{1}{x} + \log y) \text{ —— ②}$$

and

$$q = \frac{\partial z}{\partial y} = 2y + 2f'(\frac{1}{x} + \log y)(\frac{1}{y})$$

$$qy - 2y^2 = 2f'(\frac{1}{x} + \log y) \text{ —— ③}$$

From equation ② and ③, we get

$$-px^2 = qy - 2y^2$$

$$x^2p + qy = 2y^2$$

Which is a partial differential equation of the first order.

b. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$ **(8)**

Answer:

Solution (b). Here the equation is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$$

Let $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$ then

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

A.E is $m^2 + m - 6 = 0$; Put $D = m$ and $D' = 1$

$$m = -3, 2$$

$\therefore$ C.F $f_1(y-3x) + f_2(y+2x)$

How To find P.I

$$P.I = \frac{1}{D^2+DD'-6D'^2}\, y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos x$$

$$= \frac{1}{(D+3D')} \cdot \frac{1}{(D-2D')}\, y\cos x$$

$$= \frac{1}{D+3D'} \int (c-2x)\cos x\, dx$$

$\left[\because \text{Here } m=-2 \quad \cancel{y=mx+c} \quad y=mx+c \quad y=-2x+c\right.$

$$= \frac{1}{D+3D'}\left[(c-2x)\sin x - \int (-2)\sin x\, dx\right]$$

$$= \frac{1}{D+3D'}\left[(y+2x-2x)\sin x - 2\cos x\right]$$

Replace $\underline{c = y+2x}$

$$= \frac{1}{D+3D'}\left[y\sin x - 2\cos x\right]$$

Here $m=3$, $y = mx+c$, $y=3x+c$

$$= \int [(c+3x)\sin x - 2\cos x]\, dx$$

$$= (c+3x)(-\cos x) - \int 3(-\cos x)\, dx - 2\sin x$$

$$= (y-3x+3x)(-\cos x) + 3\sin x - 2\sin x$$

$$= -y\cos x + 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x$$

Hence complete solution is given by

$$z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x$$

Q.8 a. State and prove BAYE'S theorem. **(8)**

Answer:

Statement:- If $E_1, E_2, E_3 --- E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0, (i = 1,2,3 \cdots n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

Proof: Let S be the sample space of the random experiment and the events $E_1, E_2, E_3 --- E_n$ being exhaustive, then

$$S = E_1 \cup E_2 \cup E_3 \cup - \cup E_n$$

$$\therefore A = A \cap S \qquad (\because A \subset S)$$

$$= A \cap (E_1 \cup E_2 \cup E_3 \cup --- \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup ---- \cup (A \cap E_n)$$

By distributive law

The Probability P is given by

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + ---- + P(A \cap E_n)$$

$$= P(E_1) \cdot P(A/E_1) + P(E_2) P(A/E_2) + ---- + P(E_n) P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \quad \text{———} (A)$$

Now $P(A \cap E_i) = P(A)\, P(E_i/A)$

$$\therefore P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}, \text{ using (A)}$$

Hence

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

b. A can hit a target 4 times in 5 shots, B 3 times in 4 shots, C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit? **(8)**

Answer:

Probability of A's hitting the target $= \frac{4}{5}$

Probability of B's hitting the target $= \frac{3}{4}$

Probability of C's hitting the target $= \frac{2}{3}$

For at least two hits

(i) A, B, C all hit the target, the probability for which is

$$\cancel{\frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right)} = \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{24}{60}.$$

(ii) A, B hit the target and C misses it, the probability for which is

$$\frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{12}{60}$$

(iii) A, C hit the target and B misses it, the Prob for which is

$$\frac{4}{5} \times \left(1 - \frac{3}{4}\right) \times \frac{2}{3} = \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} = \frac{8}{60}$$

(iv) B, C hit the target and A misses it, the probability for which is

$$\left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{6}{60}$$

Q.9 a. Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) At least one boy (iii) No girl (iv) At most two girls? **(8)**
Assume equal probability for boys and girls

Solution 9(a).

Let p and q be the probability of having boys and girls

Since, the probability for boys and girls are equal i.e.

$$P = q = \frac{1}{2}$$

Here, n = 4 and N = 800, the binomial distribution is

$$800\left(\frac{1}{2}+\frac{1}{2}\right)^4$$

(i). the expected number of families having 2 boys and 2 girls

$$= 800\ {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 800\times 6\times\frac{1}{16} = 300$$

(ii). the expected number of families having at least one boy

$$= 800\left[{}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)+{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2+{}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3+{}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800\times\frac{1}{16}[4+6+4+1] = 750$$

(iii). the expected number of families having no girl i.e. having 4 boys

$$= 800\left[{}^4C_4\left(\frac{1}{2}\right)^4\right] = 800\times\frac{1}{16}\times 1 = 50$$

(iv). the expected number of families having at most two girls i.e. having at least 2 boys

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2+{}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3+{}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3+{}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800\times\frac{1}{16}[6+4+1] = 550$$

b. If there are 3 misprints in a book of 1000 pages find the probability that a given page will contain
(i) no misprint (ii) more than 2 misprints **(8)**

Solution (b). Here, total number of pages =1000
No of misprints are 3

Let the probability of misprints be p, then p = $\frac{3}{1000} = 0.003$

Let n =1, then m = np = $1 \times 0.003 = 0.003$

Now, Poisson distribution for 'r' outcome is

$$P(r) = \frac{e^{-m} m^r}{r!} \quad \text{------------------------ (i)}$$

(i). no misprint i.e. r = 0, putting the value of r in equation (i), then

$$P(0) = \frac{e^{-m} m^0}{0!} \quad \text{i.e.} \quad \frac{e^{-m}}{1}, \text{ here } m = 0.003, \text{ then } \quad e^{-(0.003)} = 0.997$$

Hence the probability that a page will not contain with no error is 0.997

(ii). more than two misprints i.e. r >2, i.e. r = 3,putting the value of r in equation (i), then

$$P(3) = \frac{e^{-m} m^3}{3!} \quad \text{i.e.} \quad \frac{e^{-(0.003)} (0.003)^3}{3!} = 0.0000000045$$

Hence the probability that a page will not contain more then two error is 0.0000000045

TEXT BOOK

I. Higher Engineering Mathematics –Dr. B.S.Grewal, 41st Edition 2007, Khanna Publishers, Delhi.