Q.2a. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$

Q.2 Solution (a) (7)

Let $f(x,y,z) = x^2+y^2+z^2$ and —— (i)

$\phi(x,y,z) = xyz - a^3$ —— (ii)

Now, Lagrange's method

MODERATION-I

$L(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(yz) = 0$ —— (1)

$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(xz) = 0$ —— (2)

$\frac{\partial L}{\partial z} = 0 = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} \Rightarrow 2z + \lambda(xy) = 0$ —— (3)

On multiplying (I) by x (II) by y and (III) by z, then

$2x^2 + \lambda(xyz) = 0$ —— (A)

$2y^2 + \lambda(xyz) = 0$ —— (B)

$2z^2 + \lambda(xyz = 0$ —— (C)

from (A) and (B)

$2x^2 - 2y^2 = 0 \Rightarrow x = \pm y$

from (B) and (C) $\Rightarrow y = \pm z$

Hence $x = y = z$

from relation

$xyz = a^3 \Rightarrow x^3 = a^3 \Rightarrow x = a$

hence $x = a, y = a, z = a$

Hence, the minimum value of (8)

$f(x,y,z) = 3a^2 \Rightarrow a^2 + a^2 + a^2$

b. Evaluate $\int_0^1 \frac{x^\alpha - 1}{\log x} dx,\ \alpha \ge 0,$ **by using the method of differention under the sign of integration**.

solution (b):

Let $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx, \quad \alpha \geq 0$ ——— (1)

Here α is parameter

Diff. (1) w.r.t 'α' on both sides

$$F'(\alpha) = \frac{d}{d\alpha} \int_0^1 \frac{x^\alpha - 1}{\log x} dx$$

$$= \int_0^1 \frac{\partial}{\partial \alpha} \frac{x^\alpha - 1}{\log x} dx$$

$$= \int_0^1 \frac{x^\alpha \log x}{\log x} dx$$

$$F'(\alpha) = \int_0^1 x^\alpha dx \quad \text{——— (2)}$$

Integrating w.r.t 'x' on B.H.S.

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

$$F'(\alpha) = \frac{1}{\alpha+1}$$

on integrating w.r.t to 'α'

$$F(\alpha) = \log(\alpha+1) + c \quad \text{——— (3)}$$

9)

when $x=0$, initially, $F(0)=0$

Similarly, $x=0$ finally, from (iii)

$$F(0)=\log(0+1)+c \Rightarrow 0=c \Rightarrow c=0$$

Hence Equation (iii) becomes

$$F(x)=\log(1+x)+c$$

or

$$F(x)=\log(1+x)$$ Ans.

Q.3a. Expand $f(x)=x^3$ *as* a Fourier series in the interval $-\pi < x < \pi$.

Q.3 solution (a)

Here, $f(x)=x^3$ and $-\pi x<\pi$

The function $f(x)$ is add function in $-\pi$ to π, so a_0 and a_n become zero. then remaining Fourier series is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \quad \text{———} \quad (4)$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$=\frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right)-3x^2\left(\frac{-\sin nx}{n^2}\right)+6x\left(\frac{\cos nx}{n^3}\right)-6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n}+\frac{3\pi^2\sin n\pi}{n^2}+\frac{6\pi\cos n\pi}{n^3}-\frac{6\sin n\pi}{n^4}-0\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^2} \right]$$ (10)

$$= 2(-1)^n \left[-\frac{\pi^2}{n} + \frac{6}{n^3} \right]$$

$$x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x + \cdots \right]$$

b. Obtain the half range cosine series for $\sin\left(\frac{\pi x}{l}\right)$ **in the range** $0 < x < l$.

Solution (b). Ans.

Let $f(x) = \sin\left(\frac{\pi x}{l}\right)$ and $0 < x < l$

we know that half range cosine is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\pi/l} \right]_0^l$$

$$= -\frac{2}{\pi}\left[\cos\pi - 1\right] = \frac{4}{\pi}$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

(11)

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\pi x/l}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\pi x/l}{(n-1)\pi/l} \right]_0^l$$

$$= \frac{1}{\pi}\left[\left\{ -\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} \right\} - \left\{ -\frac{1}{n+1} + \frac{1}{n-1} \right\} \right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

When n is odd

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] = 0$$

when n is even

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= -\frac{4}{\pi(n+1)(n-1)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \cdots \right]$$

Q.4a. Find the Fourier sine transform of $\dfrac{1}{x(x^2+a^2)}$.

Solution (a). (12)

Here $f(x) = \dfrac{1}{x(x^2+a^2)}$

We know that, Fourier sine Transform of $f(x)$ is

$$F_s[f(x)] = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx = I \text{ (say)} \quad \text{— (1)}$$

Diff. (1) w.r to 's', then we get

$$\frac{dI}{ds} = \int_0^\infty \frac{x\cos sx}{x(x^2+a^2)}\,dx = \int_0^\infty \frac{\cos sx}{x^2+a^2}\,dx \quad \text{— (2)}$$

Again diff. w.r.to 's' on both sides

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x \sin sx}{x^2+a^2}\,dx = \int_0^\infty \frac{-x^2 \sin sx}{x(x^2+a^2)}\,dx$$

$$= \int_0^\infty \frac{[a^2-(x^2+a^2)]\sin sx}{x(x^2+a^2)}\,dx$$

$$= a^2\int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx - \int_0^\infty \frac{\sin sx}{x}\,dx$$

$$\frac{d^2I}{ds^2} = a^2 I - \frac{\pi}{2}$$

$$\Rightarrow (D^2-a^2)I = -\pi/2, \quad \frac{d}{ds} = D$$

A.E $m^2 - a^2 = 0 \Rightarrow m = \pm a$

CF $c_1 e^{as} + c_2 e^{-as}$

$$P.I = \frac{1}{D^2+a^2}\left(-\frac{\pi}{2}\right)$$

$$= -\frac{\pi}{2}\left(\frac{e^{0s}}{D^2+a^2}\right) = -\frac{\pi}{2}\left(\frac{1}{-a^2}\right)$$

$$P.I = \frac{\pi}{2a^2}$$

$$\therefore I = C.F + P.I = c_1 e^{as} + c_2 e^{-as} + \frac{\pi}{2a^2} \quad \text{—— (3)}$$

$$\frac{dI}{ds} = a c_1 e^{as} - a c_2 e^{-as} \quad \text{—— (4)}$$

when $s = 0$ from (1), $I = 0$

from (3), $I = c_1 + c_2 + \frac{\pi}{2a^2}$

$$\Rightarrow c_1 + c_2 + \frac{\pi}{2a^2} = 0 \quad \text{—— (5)}$$

from (4), but $s = 0$

$$a c_1 - a c_2 = \frac{\pi}{2a}.$$

or $$c_1 - c_2 - \frac{\pi}{2a^2} = 0 \quad \text{—— (6)}$$

On solving (5) and (6), we get

$$c_1 = 0, \quad c_2 = -\frac{\pi}{2a^2}$$

$$\therefore I = -\frac{\pi}{2a^2} e^{-as} + \frac{\pi}{2a^2}$$

$$F_s[f(x)] = \frac{\pi}{2a^2}\left(1 - e^{-as}\right)$$

b. Find the Z-transform of sin (3k+5)

Solution (b).

(14)

$$F(z) = \sum_{k=0}^{\infty} \sin(3k+5) z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{e^{i(3k+5)} - e^{-i(3k+5)}}{2i} z^{-k}$$

$$= \frac{1}{2i} \sum_{k=0}^{\infty} e^{i(3k+5)} z^{-k} - \frac{1}{2i} \sum_{k=0}^{\infty} e^{-i(3k+5)} z^{-k}$$

$$= \frac{1}{2i} e^{i5} \sum_{k=0}^{\infty} (e^{3i} z^{-1})^k - \frac{1}{2i} e^{-5i} \sum_{k=0}^{\infty} (e^{-3i} z^{-1})^k$$

$$= \frac{1}{2i} e^{5i} [1 + (e^{3i} z^{-1}) + (e^{3i} z^{-1})^2 + \cdots] - \frac{1}{2i} e^{-5i} [1 + (e^{-3i} z^{-1}) + (e^{-3i} z^{-1})^2 + \cdots$$

$$= \frac{e^{5i}}{2i} \frac{1}{1 - e^{3i} z^{-1}} - \frac{1}{2i} e^{-5i} \frac{1}{1 - e^{-3i} z^{-1}}$$

$$= \frac{1}{2i} \frac{e^{5i}(1 - e^{-3i} z^{-1}) - e^{-5i}(1 - e^{3i} z^{-1})}{(1 - e^{3i} z^{-1})(1 - e^{-3i} z^{-1})}$$

$$= \frac{1}{2i} \frac{(e^{5i} - e^{-5i}) - e^{2i} z^{-1} + e^{-2i} z^{-1}}{1 - e^{3i} z^{-1} - e^{-3i} z^{-1} + z^{-2}}$$

$$= \frac{\frac{e^{5i} - e^{-5i}}{2i} - z^{-1} \frac{e^{2i} - e^{-2i}}{2i}}{1 - (e^{3i} + e^{-3i}) z^{-1} + z^{-2}}$$

$$= \frac{\sin 5 - z^{-1} \sin 2}{1 - (2\cos 3) z^{-1} + z^{-2}} = \frac{z^2 \sin 5 - z \sin 2}{z^2 - 2z\cos 3 + 1}$$

$|z| > 1$ A.

Q.5a. Express $f(x) = 4x^3 - 2x^2 - 3x + 8$ in term of Lagrange polynomials.

Q.5. Solution (a) (5-(a))

Let $f(x) = 4x^3 - 2x^2 - 3x + 8$

Let $4x^3 - 2x^2 - 3x + 8 = a\,P_3(x) + b\,P_2(x) + c\,P_1(x) + d\,P_0(x)$ —— (1)

$$= a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c(x) + d(1)$$

$$= \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$4x^3 - 2x^2 - 3x + 8 = \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(-\frac{3a}{2} + c\right)x - \frac{b}{2} + d$$

Equating the coefficient of like power of x on both sides

Coff of x^3 i.e. $4 = \frac{5a}{2} \Rightarrow a = \frac{8}{5}$

Coff of x^2 i.e. $-2 = \frac{3b}{2} \Rightarrow b = -\frac{4}{3}$

Coff. of x i.e. $-3 = -\frac{3a}{2} + c \Rightarrow c = -3 + \frac{3}{2}\left(\frac{8}{5}\right)$

$= -3 + \frac{12}{5}$

$c = -\frac{3}{5}$

Constant Term

$8 = -\frac{b}{2} + d \Rightarrow d = 8 + \frac{b}{2}$

Putting the values in (1), we get $= 8 + \frac{1}{2}\left(-\frac{4}{3}\right) = \frac{22}{3}$

$$f(x) = \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{3}P_0(x)$$

b. Prove that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

Solution (b). (16)

We know that

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2\cdot 2(n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2 (n+1)(n+2)} - \cdots\right] \quad \text{---(A)}$$

Putting $n = \frac{1}{2}$ in equation (A),

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma\left(\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{2\cdot 2\left(\frac{1}{2}+1\right)} + \frac{x^4}{2\cdot 4\cdot 2^2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\frac{3}{2}}\left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}\,\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{1}{\sqrt{2x}\,\frac{1}{2}\sqrt{\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{2}{\sqrt{2\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\sin x \qquad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Q.6a. Find the real root of $x^3 - 2x - 5 = 0$, correct to three decimal places using Newton-Rapson Method.

Q.6. Solution (a): (17)

Let $f(x) = x^3 - 2x - 5 = 0$ —— (1)

and $f(2) = 8 - 4 - 5 = -1$ (−ive)

$f(2.5) = (2.5)^3 - 2(2.5) - 5$

$= 5.625$ (+ive)

Here the value of the function at $f(2)$ and $f(2.5)$ are of opposite sign, so the real roots of (1) lies between 2 and 2.5.

Since $f(2)$ is near to zero then $f(2.5)$ So 2 is the better approximate roots than 2.5.

Now, $f'(x) = 3x^2 - 2 \Rightarrow f'(2) = 12 - 12 = 10$

Let 2 be an approximate root of (1), by Newton-Raphson method

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{10} = 2.1$

$\therefore f(x_1) = f(2.1) = (2.1)^3 - 2(2.1) - 5$

$= 9.261 - 4.2 - 5 = 0.061$

$\therefore f'(x_1) = f'(2.1) = 3(2.1)^2 - 2 = 11.23$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$

$= 2.1 - \frac{0.061}{11.23}$

(18)

$= 2.1 - 0.00543 = 2.09457$

$\therefore f(x_2) = f(2.09457) = (2.09457)^3 - 2(2.09457) -$

$= 9.1893 - 4.18914 - 5$

$= 0.00016$

Now $f'(2.09457) = 3(2.09457)^2 - 2$

$= 13.16167 - 2$

$= 11.16167$

$x_3 = 2.09457 - \dfrac{f(x_2)}{f'(x_2)}$

$= 2.09457 - \dfrac{0.00016}{11.16167}$

$= 2.09457 - 0.000014$

$= 2.09456$

Here x_2 and x_3 are correct up to four decimal place.

b.Apply R.K Method of fourth order, to find an approximate value of y when x = 0.1. Given that $10\frac{dy}{dx} = x^2 + y^2, y(0) = 1.$

Solution (b) . 6-(b) (19)

Here equation is

$10\frac{dy}{dx} = x^2+y^2$ and given that

$y(0) = 1$, At $x = 0.1$

Let $\frac{dy}{dx} = \frac{x^2+y^2}{10}$, let $f(x,y) = \frac{x^2+y^2}{10}$

Here let $h = 0.1$, $x_0 = 0$ and $y_0 = 1$

Now By R.K method of four order $x_0 = 0$, $h = 0.1$

$k_1 = hf(x_0, y_0) = (0.1)\, f(0, 1)$

$= (0.1)\left(\frac{0+1}{10}\right) = \underline{0.01}$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$

$= (0.1)\, f\left[0+\frac{0.1}{2}, 1+\frac{0.01}{2}\right]$

$= (0.1)\, f(0.05, 1.005)$

$= (0.1)\left[\frac{(0.05)^2+(1.005)^2}{10}\right]$

$= \underline{0.01012325}$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$

$= (0.1)\, f\left(0.05, 1+\frac{0.01012525}{2}\right)$

$= (0.1)\, f(0.05, 1.00506263)$

$= (0.1)\left[\frac{(0.05)^2+(1.00506263)^2}{\quad}\right] = \underline{0.01012651}$

$k_4 = h f(x_0+h, y_0+k_3)$

$= h f(0.1, 1+0.01012651)$

$= (0.1) f(0.1, 1.01012651)$

$= (0.1)\left[\frac{(0.1)^2+(1.01012651)^2}{10}\right]$

$= \underline{0.010303556}$

$\therefore y_{0.1} = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= 1+\frac{1}{6}[0.01+2(0.01012525)+2(0.01012651)+0.010303556]$

$= 1+0.01013451$

$= 1.01013451$

Hence y at $x=0.1$ is $\underline{1.01013451}$

Q.7 a.Solve the differential equation: $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y = x\log x$ **.**

Q.7: Solution (a) 7(a) (24)

Here $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y = x\log x$ (1)

Putting $x=e^z \Rightarrow z = \log x = \frac{dz}{dx}=\frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \Rightarrow \frac{1}{x}\frac{dy}{dz}$

$x\frac{dy}{dx} = \frac{dy}{dz} = Dy,\ D=\frac{d}{dz}$

and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Putting these values in (1), we get

$[D(D-1)-3D+5]y = ze^z$

i.e. $(D^2-4D+5)y = ze^z$

It's A.E is

$m^2-4m+5=0$

$m=\frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2\pm i$

C.F is $e^{2z}[c_1\cos z + c_2\sin z]$ — (1)

Now To find P.I.

$$P.I = \frac{ze^z}{D^2-4D+5}$$

$$= e^z\left[\frac{z}{(D+1)^2-4(D+1)+5}\right]$$

Replaced D by D+1

$$= e^z\left[\frac{z}{D^2-2D+2}\right]$$

$$= \frac{e^z}{2}\left[1+\left(\frac{D^2-2D}{2}\right)\right]^{-1} z$$

$$= \frac{e^z}{2}\left[1-\left(\frac{D^2-2D}{2}\right)+\cdots\right] z$$

(22)

$$= \frac{e^z}{2}\left[z-\frac{D^2z}{2}+\frac{2Dz}{2}+\cdots\right]$$

$$= \frac{e^z}{2}\left[z-0+1+\cdots\right]$$

$$= \frac{e^z}{2}(z+1)$$

Hence the solution is

$$y = e^{2z}[c_1\cos z + c_2\sin z] + \frac{e^z}{2}(z+1)$$

or

$$y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] + \frac{x}{2}(\log x + 1)$$

b.Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\,e^x \sin x$

Q7 Solution (b)

Here, the given diff-equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

It can be written as

$$(D^2 - 2D + 1)y = xe^x \sin x, \quad \left(\because \frac{d}{dx} = D\right)$$

Its A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F is $(C_1 + xC_2)e^x$ —— ①

$$P.I = \frac{xe^x \sin x}{D^2 - 2D + 1}$$

$$= \frac{e^x (x \sin x)}{(D+1)^2 - 2(D+1) + 1}$$

$$P.I = e^x \frac{x \sin x}{D^2 + 1 + 2D - 2D - 2 + 1}$$

201/3 (23)

$$= e^x \left[\frac{x \sin x}{D^2}\right]$$

$$= \frac{e^x}{D} \int x \sin x \, dx$$

$$= \frac{e^x}{D}\left[x(-\cos x) - \int(-\cos x)\,dx\right]$$

$$= \frac{e^x}{D}\left[-x\cos x + \sin x\right]$$

$$= e^x \int(-x\cos x + \sin x)\,dx$$

$$= e^x\left[(-x)\sin x - \int(-1)\sin x\,dx - \cos x\right]$$

$$= e^x\left[-x\sin x - \cos x - \cos x\right]$$

$$= e^x\left[-x\sin x - 2\cos x\right]$$

$$= -e^x(x\sin x + 2\cos x)$$

Its solution is

$$y = C.F + P.I$$

$$y = (C_1 + xC_2)e^x - e^x(x\sin x + 2\cos x)$$

Q.8 a. Find the rank of the matrix A, where A= $\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$**, by reducing it to normal form.** **(8)**

Q.8 Solution (a):

Here $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{pmatrix}$$

$C_2 \to C_2 - 3C_1$

$C_3 \to C_3 - 4C_1$

$C_4 \to C_4 - 2C_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$R_2 \longrightarrow R_3 \longleftrightarrow R_4$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \rightarrow C_3 - \frac{5}{7} C_2$$

$$C_4 \rightarrow C_4 - \frac{2}{7} C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{1}{7} C_2$$

$$R_3 \rightarrow -R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ whi is normal for

Hence, Rank $(A) = \underline{\underline{3}}$

b.Investigate the value of λ *and* μ **so that the equations** $2x+3y+5z=9;\ 7x+3y-2z=8;\ 2x+3y+\lambda z=\mu$ **(8)**
(i) No solution (ii) Unique solution (iii) An infinite number of solutions

Solution (b):

Here the system of equation are

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

(26)

These system of equation can be written in the matrix form i.e.

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$A \quad X = B$$

Now, Augmented matrix

$$C = [A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$$R_2 \to R_2 - \frac{7}{2}R_1$$
$$R_3 \to R_3 - R_1$$

or $$C = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & : & -\frac{47}{2} \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

Case I For No solution:

Rank A $\neq$ Rank C

$\therefore \lambda-5=0 \Rightarrow \lambda=5$ and $\mu-9\neq 0 \Rightarrow \mu\neq 9$

Case II For Unique solution:

Rank(A) = Rank (C) = Number of unknowns

$\lambda-5\neq 0 \Rightarrow \lambda\neq 5$ and $\mu\neq 9$

Q.9 a. Evaluate $\int_0^\infty\int_0^\infty x\, e^{-\frac{x^2}{y}}\, dx\,dy$

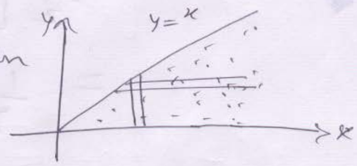
Q.9 Solution(a).

Here $\int_{x=0}^{\infty}\int_{y=0}^{x} x\, e^{-\frac{x^2}{y}}\, dx\, dy$

limits are $x=0$ to $x=\infty$

and $y=0$ to $y=x$

So region are the sadded portion



Now, To change the order of integration i.e. direction of strip i.e.

$x=y$ to ∞ and $y=0$ to ∞, then.

$$\int_0^\infty\int_0^\infty x\, e^{-\frac{x^2}{y}}\, dx\, dy = \int_{y=0}^{\infty}\int_{x=y}^{\infty} x\, e^{-\frac{x^2}{y}}\, dx\, dy$$

$$= \int_0^\infty \left[\int_{y^2}^{\infty} e^{-\frac{t}{y}}\, \frac{dt}{2}\right] dy$$

Put $x^2 = t \Rightarrow 2x\, dx = dt$ (2x)
when $x=y$, $t=y^2$
$x=\infty$, $t=\infty$

$$= \frac{1}{2}\int_0^\infty dy \left(\frac{e^{-\frac{t}{y}}}{-\frac{1}{y}}\right)_{y^2}^{\infty}$$

$$= \frac{1}{2}\int_0^\infty dy \left(-y\, e^{-\frac{\infty}{y}} + y\, e^{-\frac{y^2}{y}}\right)$$

$$= \frac{1}{2}\int_0^\infty dy \left(0 + y\, e^{-y}\right)$$

$$= \frac{1}{2}\int_0^\infty y\, e^{-y}\, dy$$

$$= \frac{1}{2}\left[\left(y\, \frac{e^{-y}}{-1}\right)_0^\infty - \int_0^\infty \frac{e^{-y}}{-1}\cdot 1\cdot dy\right]$$

$$= \frac{1}{2}\left[(0-0) + 1\left(\frac{e^{-y}}{-1}\right)_0^\infty\right]$$

$$= \frac{1}{2}\left[-\left(e^{-\infty} - e^{0}\right)\right]$$

$$= \frac{1}{2}(0+1) = \frac{1}{2}$$

b. Define Beta and Gamma functions. Prove that $\beta(m,n) = \dfrac{\gamma(m)\gamma(n)}{\gamma(m+n)}$.

Solution (b):

Beta Function: If m, n are positive, then the definite integration $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the Beta function and is denoted by $\beta(m,n)$, Hence

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m>0,\ n>0$$

(29)

Gamma Function: If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$, which is a function x is called the Gamma function and is denoted by Γn, Hence

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \quad n>0$$

Now, Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

By the definition of Gamma function

$$\Gamma(m) = \int_0^\infty e^{-z} z^{m-1}\,dz$$

Putting $z = x^2$
$dz = 2x\,dx$

$$\therefore \Gamma(m) = \int_0^\infty e^{-x^2} x^{2m-1}\,dx \qquad \text{—(1)}$$

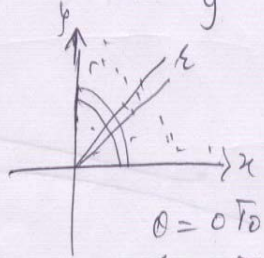
Similarly

$$\therefore \Gamma(n) = \int_0^\infty e^{-y^2} y^{2n-1}\, dy \quad \text{—— (1)}$$

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy$$

$y=$ $x=$

On changing to polar coordinate, then.



$\theta = 0$ to $\frac{\pi}{2}$

$r = 0$ to ∞

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \quad \text{—— (A)}$$

Now $2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr = \Gamma(m+n)$

and $2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta$

Put $\sin^2\theta = z$

$2\sin\theta\cos\theta \, d\theta = dz$

$z = 0$ To 1

$$= \int_0^1 (1-z)^{m-1} z^{n-1} dz$$

$$= \int_0^1 z^{n-1} (1-z)^{m-1} dz$$

$$= \beta(n\ m)$$

By symmetric property

$$= \beta(m, n)$$

From (A).

$$\Gamma(m)\Gamma(n) = \Gamma(m+n) \cdot \beta(m, n)$$

Hence $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Textbook

I. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi