Q.2 a. Let A = {Φ, b}, construct the following sets: **(8)**

(i) A – Φ

(ii) {Φ} – A

(iii) $A \cap P(A)$

Answer:

Given $A = \{\phi, b\}$, then

(i) $A - \phi = A - \{\ \} = \{\phi, b\}$

(ii) $\{\phi\} - A = \{b\}$

(iii) $P(A) = \{\phi, \{\phi\}, \{b\}, \{\phi, b\}\}$

then $A \cap P(A) = \phi$.

b. Prove that (i) $(A \cap B) \cup (A \cap \sim B) = A$

(ii) $A \cap (\sim A \cup B) = A \cap B$

Ans.(b): (i) By distributive property of sets 4 + 4

$$(A \cap B) \cup (A \cap \sim B) = A \cap (B \cup \sim B)$$
$$= A \cap (U)$$
$$= A$$

Here U is the universal set.

(ii) $A \cap (\sim A \cup B) = (A \cap \sim A) \cup (A \cap B)$
$$= \phi \cup (A \cap B)$$
$$= (A \cap B)$$

Q.3a. (i) Given the value of $p \rightarrow q$ is true. Determine the value of $\sim p \vee (p \leftrightarrow q)$.

(ii) Is the statement tautology?

$(p \wedge (p \rightarrow q)) \rightarrow q$

Q.3 (a) (i) Given the value of p → q is true. Determine the value of $\sim p \vee (p \leftrightarrow q)$. **(8)**

Ans.(a): (i) Given p→q is true. Then $\sim p \vee (p \leftrightarrow q)$ can be written as: 5

$$\sim p \vee (p \leftrightarrow q) = \sim p \vee ((p \rightarrow q) \wedge (q \rightarrow p))$$

$$= \sim p \vee (T \wedge (q \rightarrow p))$$

$$= \sim p \vee (q \rightarrow p)$$

$$= \sim p \vee (\sim q \vee p) \quad [\, p \rightarrow q \equiv \sim p \vee q \,] \text{ (logically equivalent)}$$

$$= (\sim p \vee p) \vee q = T \vee q = \mathrm{T} \quad \text{(by commutative \& associative prop)}$$

(ii) Is the statement tautology?

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

(ii) Draw the truth table for $(p \wedge (p \rightarrow q)) \rightarrow q$ as: 3

p	q	p→q	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Hence, the given preposition is a tautology.

b. Define logical equivalence. Construct the truth tables for the propositions $(P \vee Q) \wedge \sim(P \wedge Q)$ and $(P \wedge \sim Q) \vee (\sim P \vee Q)$.
Also deduce that whether the above pairs are equivalent.

Ans.(b): Logical Equivalence: Two prepositions P(p, q, ∨, ∧, ...) and Q(p, q, ∨, ∧, ...) are said to be logically equivalent if they have the same truth table.

Truth table for $(P \vee Q) \wedge \sim (P \wedge Q)$ and $(P \wedge \sim Q) \vee (\sim P \vee Q)$:-

P	~P	Q	~Q	P∨Q	~(P∧Q)	$(P \vee Q) \wedge \sim (P \wedge Q)$	(P∧~Q)	(~P∨Q)	$(P \wedge \sim Q) \vee (\sim P \vee Q)$
T	F	T	F	T	F	F	F	F	F
T	F	F	T	T	T	T	T	F	T
F	T	T	F	T	T	T	F	T	T
F	T	F	T	F	T	F	F	F	F

The results in red shows the two prepositions are logically equivalent as they have same truth table.

Q.4 a. Define quantifiers. Negate the statement:
For all real x, if x > 3, then x^2 > 9.

Ans.(a): Quantifiers:- There are two types of quantifiers

Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)p(x) \qquad \text{or} \qquad \forall x\, p(x) \tag{4.1}$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$". The symbol

$$\forall$$

which reads "for all" or "for every" is called the *universal quantifier*. The statement (*4.1*) is equivalent to the statement

$$T_p = \{x : x \in A,\ p(x)\} = A \tag{4.2}$$

that is, that the truth set of $p(x)$ is the entire set A.

Q_1: If $\{x: x \in A, p(x)\} = A$ then $\forall x\, p(x)$ is true; otherwise, $\forall x\, p(x)$ is false.

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)p(x) \quad \text{or} \quad \exists x,\ p(x) \tag{4.3}$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$". The symbol

$$\exists$$

which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*. Statement (*4.3*) is equivalent to the statement

$$T_p = \{x: x \in A, p(x)\} \neq \varnothing \tag{4.4}$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\, p(x)$, that is, $p(x)$ preceded by the quantifier $\exists$, does have a truth value. Specifically:

Q_2: If $\{x: p(x)\} \neq \varnothing$ then $\exists x\, p(x)$ is true; otherwise, $\exists x\, p(x)$ is false.

Negation of Quantified statements: The given statement "For all real x, if x > 3, then $x^2 > 9$" can be written using quantifiers as:

$$\forall x \in R(p(x) \to q(x))$$

Where p(x) : x > 3 and q(x) : $x^2 > 9$

It's negation is written as:

$$\sim[\forall x \in R(p(x) \to q(x))] = \exists x \in R(\sim(p(x) \to q(x))) \quad \text{(using DeMorgan's Law)}$$

$$= \exists x \in R(\sim(\sim p(x) \vee q(x)))$$

$$= \exists x \in R(p(x) \wedge q(x))$$

Thus it is "There is at least one real x such that x > 3 and $x^2 > 9$".

b. Define the validity of the following argument: (8)
"If Ram runs for office, he will be selected. If Ram attends the meeting, he will run for office. Either Ram will attend the meeting or he will go to London. But Ram cannot go to London. Thus Ram will be selected."

Ans.(b): Let p: Ram runs for office, q: He will be selected, r: Ram attends the meeting

s: He will go to London

The argument is: $p \rightarrow q, r \rightarrow p, r \vee s, \sim s$ |----- q

$r \vee s = s \vee r$	(by commutative property)
$\sim s \rightarrow r$	(because $p \rightarrow q \equiv \sim p \vee q$)
$\sim s$	(given hypothesis)
r	(by rule of detachment i.e; p, $p \rightarrow q$ gives q)
$r \rightarrow p$	(given)
p	(by rule of detachment i.e; p, $p \rightarrow q$ gives q)
$p \rightarrow q$	(given hypothesis)
Hence q	(by rule of detachment i.e; p, $p \rightarrow q$ gives q)

Thus the given argument is valid.

Q.5 a. Define Cartesian product on sets. For the given sets X = {1, 2}, Y = {a, b, c} and Z = {c, d}, find (X×Y) ∩ (X×Z).

Ans.(a): **<u>Cartesian Product:</u>** Let A and B be two non-empty sets. Then the Cartesian product A×B contains all the ordered pair (a, b) where the first element of the ordered pair is from the first set and the second element is from the second set. That is $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ 2

Given X = {1, 2}, Y = {a, b, c} and Z = {c, d}, then

$$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$ 2

$$X \times Z = \{(1, c), (1, d), (2, c), (2, d)\}$$ 2

Then $(X \times Y) \cap (X \times Z) = \{(1, c), (2, c)\}$ as these ordered pairs are common in both X×Y and X×Z. 2

b. Let a and b be positive integers, and suppose A is defined recursively as follows:

$$A(a,b) = \begin{cases} 0 & , if\ a < b \\ A(a-b,b)+1, & if\ b \le a \end{cases}$$

(i) Find: (i) A (2, 5), (ii) A(12, 5).

(ii) What does this function A do? Find A (5861, 7).

Ans.(b): (a) (i) $A(2, 5) = 0$ since $2 < 5$ {using the given definition of $A(a, b)$} 2

(ii) $A(12, 5) = A(12 - 5, 5) + 1$; as $12 > 5$ 2

$= A(7, 5) + 1$

$= \{A(7 - 5, 5) + 1\} + 1$; as $7 > 5$

$= A(2, 5) + 2 = 0 + 2 = 2$

(b) Each time b is subtracted from a, the value of A is increased by 1. Hence A (a, b) finds the quotient when a is divided by b. 2

Thus A (5861,7) = 837. 2

Q.6a. **Let R be binary relation on the set of all strings of 0s and 1s such that R = {(a, b) | a and b are strings that have same number of 0s}:**

(i) Is R reflexive?
(ii) Is R symmetric?
(iii) Is R transitive?
(iv) Is R a partial order relation?

(8)

Ans.(a): Given R = {(a, b) | a and b are strings that have same number of 0s}

(i) Reflexive: R is reflexive if for all a in R, aRa. Hence the given relation is reflexive. 2

(ii) Symmetric: R is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ 2

Let $(a, b) \in R \Rightarrow$ a and b are strings that have same number of 0's

⇨ b and a are the strings that have same number of 0's

⇨ $(b, a) \in R$

⇨ R is symmetric

(iii) R is transitive: Let $(a, b) \in R$ and $(b, c) \in R$ 2

⇨ String a and b have same no of 0's also string b and c have same no of 0's

⇨ String a and c have same no of 0's

⇨ $(a, c) \in R$

⇨ R is transitive.

(iv) Partial order relation: A relation R is called partial order if R is (i) reflexive, (ii) antisymmetric that is: if $(a, b) \in R$ and $(b, c) \in R \Rightarrow a = b$, and (iii) transitive. 2

Given relation R is reflexive and transitive but not anti-symmetric.

For eg. Let a = 0001 and b = 10010, both having same no of 0's but $a \neq b$.

Hence the given relation is **not** a partial order relation.

b. Prove that if a and b are elements in a bounded distributive lattice and if a has a compliment a', then

(i) $a \vee (a' \wedge b) = a \vee b$

(ii) $a \wedge (a' \vee b) = a \wedge b$

Ans.(b) Given that the lattice is bounded and a has a compliment a'.

=> $a \vee a' = I$ and $a \wedge a' = O$

(i) $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$ by distributivity 4

$= I \wedge (a \vee b)$

$= (a \vee b)$

(ii) $a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$ by distributivity 4

$= O \vee (a \wedge b)$

$= (a \wedge b)$

Q.7 a. Let f : R → R be a function defined as, f(x) = |x|. Show that f is neither one-one nor onto function.

Ans.(a) Given f : R → R as f(x) = |x|.

f is not one-one: clearly f is not a one-one function as f(x) = |x| = x =|-x| = f(-x) but x ≠ -x.

For eg. |2| = 2 and |-2| = 2 but 2 ≠ -2 4

f is not onto: A function is said to be onto if Range(f) = co domain of (f). Here the co-domain is R (real line) and range of function f(x) = |x| for all x in R.

Range(f) = {|x|; for all x in R} = {+ve x; for all x in R as |x| gives +ve values} = R^+ 4

Clearly Range(f) = R^+ ≠ co-domain(f) = R.

Hence f is neither one-one nor onto function.

b. Define composite functions. Let f : R → R be a function given as, f(x) = 2-x^2 and g : R^+ → R^+ be given as $g(x) = \sqrt{x}$, where R^+ is the set of non-negative real numbers. Compute fog(x) and gof(x).

Ans.(b) Composite functions:- Let A, B and C be three sets. Let f: A → B is defined as f(x) = y and g: B → C is defined as g(y) = z. Then a function (gof) : A → C is defined as

(gof)(x) = {z | for y in B there exist f(x) = y and g(y) = z) is called composite function from A to C.

Now, given $f : R \rightarrow R$ be a function given as, $f(x) = 2 - x^2$ and $g : R^+ \rightarrow R^+$ given as $g(x) = \sqrt{x}$,

Then $(gof)(x) = g(f(x)) = g(2 - x^2) = \sqrt{(2 - x^2)}$ for all $x \leq 1$as for $x \geq 2$, $(2 - x^2) < 0$ and then squar root cannot be obtained. Thus $(gof)(x)$ for all x in R cannot be obtained.

Now, $(fog)(x) = f(g(x)) = f(\sqrt{x}) = 2 - (\sqrt{x})^2 = 2 - x$ for all x in R^+.

Q.8 a. If Z_n denotes the set of integers {0, 1, 2, …., n-1} and * be binary operation on Z_n such that a*b = the remainder of ab divided by n,

(i) Construct the table for the operation * for n = 4

(ii) Show that (Z_n, *) is a semi-group for n = 4.

(8)

Ans.(a): (i) Table for operation * for n = 4:

$*_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

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(ii)**Semi group:** An algebraic structure (S, *) is called a semi-group if (i) S is closed with respect to the binary operation and (ii) * is associative i.e. for all a, b, c in S $a*(b*c) = (a*b)*c$.

From the table it is clear that operation $(Z_4, *_4)$ is closed as all the elements in the table are from set $Z_4 = \{0, 1, 2, 3\}$

Associative: consider 3 elements from Z_4 as 1, 2 and 3.

Then $1*_4(2*_43) = 1*_42 = 2$ and

$(1*_42)*_43 = 2*_43 = 2$

4

Hence $(Z_4, *_4)$ is associative and a semi-group.

b. Let (R, +) be the additive group of real numbers and (R^+, ×) be a multiplicative group of positive real numbers. Prove that $f : R \rightarrow R^+$, defined by $f(x) = e^x$, for all x in R is an isomorphism from (R, +) to (R^+, ×).

Ans.(b): Two groups $(G_1, +)$ and $(G_2, *)$ are said to be isomorphic if there exist a function $f : G_1 \rightarrow G_2$ such that (i) f is one-one, (ii) f is onto and (iii) $f(a + b) = f(a) * f(b)$.

2+6

Given that (R, +) is the additive group of real numbers and $(R^+, \times)$ is a multiplicative group of positive real numbers and $f : R \rightarrow R^+$, defined by $f(x) = e^x$, for all x in R.

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(i) f is one-one: Let $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 \log_e e = x_2 \log_e e \Rightarrow x_1 = x_2$. Hence f is one-one.

(ii) f is onto: Let $y \in R^+$ such that $f(x) = y \Rightarrow e^x = y \Rightarrow x = \log_e y$. Since y is a +ve real number $\log_e y$ will exist and be a real number.

Thus for all $y \in R^+$ there exist $x \in R$ such that $f(x) = y \Rightarrow$ f is onto function.

(iii) Now let $x_1, x_2 \in R$, then $f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1} \times e^{x_2} = f(x_1) \times f(x_2)$.

Thus (R, +) and $(R^+, \times)$ are isomorphic to each other under the specified function.

Q.9 a. The generating function of an encoding function $E : Z_2^3 \rightarrow Z_2^6$ is given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(i) Find the code words assigned to 110 and 010. **4**
(ii)Obtain associated parity-check matrix. **2**
(iii)Hence decode the received word 110110. **2**

Ans.(a) We note that G = [I_3 / A] where,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(i) We find that

$$[E(110)] = [1 \quad 1 \quad 0] \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = [1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1]$$

$$[E(010)] = [0 \quad 1 \quad 0] \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = [0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1]$$

Thus the required code words are

$E(110) = 110101$ and $E(010) = 010011$.

(ii) The parity-check matrix associated with G is given by H = [A^T / I_3]

$$H = [A^T / I_3] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(iii) For w = 110110, the syndrome of w is

$$H[1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0]^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We observe that this matrix is identical with the second column of H. Therefore, w
change the second component in w (from 1 to 0) to get the word c = 100110.
The first three components of this code word gives the original message r =100.

b.Let n be an integer satisfying n > 1. Then prove that the ring Z_n of congruence classes of integer modulo n is an integral domain if and only if n is a prime number.

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Ans.9(b): First we show that Z_n is an integral domain only if n is a prime number. Suppose that n is not a prime number. Then n = rs, where r and s are integers satisfying $0 < r < n$ and $0 < s < n$. Let [r] and [s] denote the congruence classes of r and s modulo n. Then [r] and [s] are non-zero elements of Z_n, and [r][s] = [rs] = [n] = [0]. It follows that if n is not a prime number then Z_n is not an integral domain.

We must show also that if n is a prime number then Z_n is an integral domain. Let α and β be elements of Z_n. If $\alpha \neq [0]$ and $\beta \neq [0]$ then there exist integers r and s satisfying $0 < r < n$ and $0 < s < n$ such that $\alpha = [r]$ and $\beta = [s]$. Clearly since r and s satisfying $0 < r < n$ and $0 < s < n$, rs is not divisible by any prime number p, and thus $\alpha\beta = [r][s] = [rs] \neq 0$. We have thus shown that if n is a prime number then the product of any two non-zero elements of Z_n is non-zero. We conclude that if n is a prime number then Z_n is an integral domain, as required.

Text Book

1. Discrete Mathematical Structures, D. S. Chandrasekharaiah, Prism Books Pvt. Ltd., 2005