Q.2 a. If $y = X^{X^{X^{\cdots\cdots\infty}}}$ **, then prove that** $X\frac{dy}{dX} = \frac{y^2}{(1 - y\log_e X)}$

Ans. we have $y = X^{X^{X^{\cdots\cdots\infty}}} = X^y$ $\left[\because X^{X^{X^{\cdots\cdots\infty}}} = y\right]$

by taking ℓog of both sides, we get

$\ell og\, y = y\ \ell og\, x$

differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + (\ell og x)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \ell og x\right) = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{1 - y\ell og x}$$

Hence Proved

b. Find the equation of the tangent to the curve $x^2 + 2y = 8$ **which is perpendicular to the line** $x - 2y + 1 = 0$**.**

Ans. The given curve is $x^2 + 2y = 8$ ―― ― ― ― ― ― ― ― ― ― ― ― ― ― ― ― ― ― (i)

Differentiating w.r.t. 'x' we get

$$2x + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -x$$

and the given line is $x - 2y + 1 = 0$

Differentiating w.r.t. 'x' we get

$$1 - 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Since the tangent is perpendicular to the line therefore, (slope of tangent) (slope of line) = -1

$$\Rightarrow (-x)\left(\frac{1}{2}\right) = -1 \Rightarrow x = 2$$

Now, we have to find y co-ordinate when x = 2 on putting x = 2 in (i), we get

$(2)^2 + 2y = 8 \Rightarrow 2y = 8 - 4 = 4 \Rightarrow y = 2$

$\therefore$ Equation of tangent to (i) at the point (2, 2) is,

$y - 2 = -2(x - 2)$

$\Rightarrow y - 2 = -2x + 4$

$\Rightarrow 2x + y = 6$

Q.3 a. Evaluate $\int e^{-x}.\cos x dx$

Ans. Let I = $\int e^{-x}.\cos x dx$, then

$$\overset{I}{e^{-x}} \quad \overset{II}{\cos x}$$

$= e^{-x}\sin x - \int -e^{-x}.\sin x dx$

$= e^{-x}\sin x + \int e^{-x}.\sin x dx$

I II

$= e^{-x}\sin x + e^{-x}(-\cos x) - \int(-e^{-x})(-\cos x)dx$

$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x dx$

or I = $e^{-x}\sin x - e^{-x}\cos x$ - I

or 2I = $e^{-x}(\sin x - \cos x)$

or I = $\frac{e^{-x}}{2}(\sin x - \cos x)$

b. Evaluate $\int_1^2 \frac{1}{x(1+x^2)}dx$

Ans. Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$, then

1 = A (1 + x^2) + (B x + C) x ____________ (i)

Putting x = 0 is (i), we get A = 1. Comparing the coefficients of x^2 and x, we get,

$A + B = 0$ and C = 0 $\Rightarrow$ B = -1 and C = 0 $[\because A = 1]$

$\therefore \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

So, $\int_1^2 \frac{1}{x(1+x^2)}dx = \int_1^2 \frac{1}{x}dx - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}dx$

$= [\ell og x]_1^2 - \frac{1}{2}[\ell og(1+x^2)]_1^2$

$= (\ell og 2 - \ell og 1) - \frac{1}{2}[\ell og 5 - \ell og 2]$

$= \ell og 2 - \frac{1}{2}\ell og 5 + \frac{1}{2}\ell og 2$

$= \frac{3}{2}\ell og 2 - \frac{1}{2}\ell og 5$

Q.4 a. Find the matrix A satisfying the equation $\begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} A \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans. Let $|B| = \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}$ and $|C| = \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix}$, then

$|B| = \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = 6 - 3 = 3 \neq 0$ and

$|C| = \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = 10 - 9 = 1 \neq 0$

So, B and C are invertible matrices. The given matrix equation is

BAC = I

on pre- multiplying by B^{-1} and post multiplying by C^{-1}, we get,

$\Rightarrow B^{-1} (BAC) C^{-1} = B^{-1} I C^{-1}$

$\Rightarrow (B^{-1}B) A (C C^{-1}) = B^{-1}.C^{-1}$

$\Rightarrow I A I = B^{-1}.C^{-1}$

$\Rightarrow A = B^{-1}. C^{-1}$ ________________ (i)

Matrix of co- factors of B = $\begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix}$

Adjoint B = $\begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}$ Adj B = $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix}$ ______________ (ii)

Matrix of co-factors of C = $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Adjoint C = $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$C^{-1} = \frac{1}{|C|}$ Adj C = $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ ____________________ (iii)

on putting the value of $B^{-1}$from (ii) and C^{-1} from (iii) in (i), we get

$\therefore A = B^{-1}.C^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 6+3 & -9+5 \\ -6-6 & 9+10 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -14 \\ -12 & 19 \end{bmatrix}$

$= \begin{bmatrix} 3 & -14/3 \\ -4 & 19/3 \end{bmatrix}$

b. Solve the following set of equations by using Cramer's rule

$$2x - y + 3z = 9, x + y + z = 6, x - y + z = 2$$

Ans. Let $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ = -2

$$\Delta x = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -2$$

$$\Delta y = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -4$$

$$\Delta z = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = -6$$

Now, $x = \frac{\Delta x}{\Delta} = \frac{-2}{-2} = 1$

$y = \frac{\Delta y}{\Delta} = \frac{-4}{-2} = 2$

$z = \frac{-6}{-2} = 3$

x = 1, y = 2, z = 3

Q.5 a. Solve $x^2 dy + y(x + y)dx = 0$

Ans. The given equation is $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$ ____________ (i)

which is a homogeneous differential equation of first order, putting y = v x and $\frac{dy}{dx}$ = v + x $\frac{dv}{dx}$, equation (i) yields

$$v + x\frac{dv}{dx} + \frac{vx(x+vx)}{x^2} = 0$$

or $x\frac{dv}{dx} + v^2 + 2v = 0$

or $\dfrac{dv}{v^2+2v}+\dfrac{dx}{x}=0$

$$\frac{1}{2}\left(\frac{1}{v}+\frac{1}{v+2}\right)dv+\frac{dx}{x}=0$$

on integrating the required solution is

$$\frac{1}{2}\ [\ell og v-\ell og(v+2)]+\ell og x=\ell og c$$

$$\ell og\left(x\sqrt{\frac{v}{v+2}}\right)=\ell og c$$

or $x\sqrt{\dfrac{y/x}{y/x+2}}=c$, as $v=\dfrac{y}{x}$

or $x\sqrt{\dfrac{y}{y+2x}}=c$

$x^2y = a(y + 2x)$ where $c^2 = a$

b. Solve $(1+y^2)dx+xdy=\tan^{-1}y\,dy$

Ans. $(1+y^2)dx = (\tan^{-1}y - x)\,dy$

$$\Rightarrow\quad (1+y^2)\frac{dx}{dy}=\tan^{-1}y-x$$

$$\Rightarrow\quad \frac{dx}{dy}=\frac{\tan^{-1}y}{1+y^2}-\frac{x}{1+y^2}$$

$$\Rightarrow\quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow\quad \text{I. F.}=e^{\int\frac{1dx}{1+y^2}}$$

$$=e^{\tan^{-1}y}$$

So solution of the given differential equation is,

$$= x\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} . dy + c$$

$$\Rightarrow x\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y . e^{\tan^{-1} y} . dy}{1+y^2} + c$$

Put $\tan^{-1} y = t$

$$\Rightarrow \frac{1}{1+y^2} dy = dt$$

$$\Rightarrow x\, e^{\tan^{-1} y} = \int t e^t dt + c$$

$$\Rightarrow x\, e^{\tan^{-1} y} = t\, e^t - \int e^t dt + c$$

$$\Rightarrow x\, e^{\tan^{-1} y} = t\, e^t - e^t + c$$

$$\Rightarrow x\, e^{\tan^{-1} y} = e^t (t - 1) + c$$

$$\Rightarrow x\, e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y + 1) + c$$

$$\Rightarrow (\tan^{-1} y - 1) + c\, e^{\tan^{-1} y} = x$$

Q.6 a. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-1/2}$ is $\frac{(2r)!}{(r!)^2}$

Ans. We know that the general term of the expansion

$(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2)...............(n+1-r).x^r}{r!}$

Putting $n = -\frac{1}{2}$ and $-4x$ for x, we get

$$T_{r+1} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right).......\left[-\left(\frac{2r-1}{2}\right)\right]}{r!} \; (-1)^r . (2^2)^r . x^r$$

$$= (-1)^r(-1)^r \frac{1.3.5.7......(2r-1)}{2^r \times r!}.2^{2r}.x^r$$

$$= \frac{1.3.5.7.......(2r-1)}{r!}2^r.x^r$$

Multiplying and dividing by 2.4.6...2r, we have

$$= \frac{1.2.3.4.......(2r-1)^{2r}}{[2.4.6.8...2r]r!}2^r.x^r = \frac{2r!}{2^r(1.2.3..r)r!}2^r.x^r$$

$$= \frac{2r!}{r!r!}x^r = \frac{2r!}{(r!)^2}x^r$$

Hence, the coefficient of x^r is $\frac{2r!}{(r!)^2}$ Proved

b. Find three number in A.P. whose sum is 21 and their product is 315.

Ans. Let three numbers is A.P. be,

a – d, a, a + d ____________ (i)

Their sum a – d + a +a+ d = 21

$\therefore$ 3a = 21 $\Rightarrow$ a = 7

Now product of 3 numbers = (a – d) a (a + d)

$\therefore$ (a – d) . a (a + d) = 315

Putting the value of a, we get

(7-d) 7 (7 + d) = 315

$\Rightarrow (49 - d^2) = \frac{315}{7} = 45$

$\Rightarrow d^2 = 49-45 = 4$

$\Rightarrow d = \pm\ 2$

Case I- When a = 7, d = 2 putting in (i)

$\therefore$ Numbers are 7-2, 7, 7+2 i.e. 5, 7, 9

Case II When a = 7, d = -2 putting in (i)

$\therefore$ Numbers 7+2, 7, 7-2 is 9, 7, 5

$\therefore$ Required three numbers are 5, 7, 9, or 9, 7, 5

Q.7 a. If A, B, C are the angles of a triangle, then prove that, tan 2A+ tan 2B + tan2C = tan 2A.tan 2B.tan 2C

Given that $A + B + C = \pi = 180^o$

then $2A + 2B + 2C = 360^o$

or $2A + 2B = 360^o - 2C$

Taking tangent of both sides

$\tan(2A + 2B) = \tan(360^o - 2C)$

$\frac{\tan 2A + \tan 2B}{1 - \tan 2A.\tan 2B} = -\tan 2C$

Cross-multiplying

$\therefore \tan 2A + \tan 2B = -\tan 2C + \tan 2A.\tan 2B.\tan 2C$

Transpasing,

$\tan 2A + \tan 2B + \tan 2C = \tan 2A.\tan 2B.\tan 2C$

Hence Proved.

b. Prove that, $\sin 10^\circ.\sin 50^\circ.\sin 60^\circ.\sin 70^\circ = \frac{\sqrt{3}}{16}$

Ans. We have,

LHS = $\sin 10^\circ.\sin 50^\circ.\sqrt{3}/2.\sin 70^\circ$

$= \frac{\sqrt{3}}{2}.\frac{1}{2}\left[2\sin 50^o.\sin 10^o\right].\sin 70^o$

$= \frac{\sqrt{3}}{4}\left[\cos(50^o - 10^o) - \cos(50^o + 10^o)\right].\sin 70^o$

$= \frac{\sqrt{3}}{4}\frac{1}{2}\left[2\sin 70^o.\cos 40^o - 2\sin 70^o \cos 60^o\right]$

$= \frac{\sqrt{3}}{8}\left[\sin(70^o + 40^o) + \sin(70^o - 40^o) - 2\sin 70^o \frac{1}{2}\right]$

$= \frac{\sqrt{3}}{8}\left[\sin 110^o + \sin 30^o - \sin 70^o\right]$

$= \frac{\sqrt{3}}{8}\left[\sin(180^o - 70^o) + \frac{1}{2} - \sin 70^o\right]$

$= \frac{\sqrt{3}}{8}\left[\sin 70^o + \frac{1}{2} - \sin 70^o\right]$

$= \frac{\sqrt{3}}{8} \times \frac{1}{2} = \sqrt{\frac{3}{16}}$ = RHS

Hence Proved

Q.8 a. Find the equation of the straight lines through the point $(2,-1)$ and making an angle of 45° with the line 6x+5y-1 = 0.

Ans. Equation of the straight line through $(2,-1)$ and having slope m is
y + 1 = m(x + 2) ______________(i)

(i) Makes an angle 45^o with the line
6x + 5y – 1 = 0 ____________(ii)

Slope of (ii) in - $\frac{6}{5}$

$$\tan 45^o = \pm \frac{m+\frac{6}{5}}{1-\frac{6m}{5}} \quad \text{or} \quad 1 = \pm \frac{5m+6}{5-6m}$$

taking + ve sign,

5-6m = + (5m+6) $\Rightarrow$ m = - $\frac{1}{11}$

Taking -ve sign,

5-6m = - (5m + 6) $\Rightarrow$ m = 11

Putting the values of m in (i) we get y + 1 = - $\frac{1}{11}$(x – 2) and

y + 1 = 11 (x – 2)

or x +11y + 9 = 0 and 11x – y -23 = 0 are the required eqn. of the & line.

Ans. x + 11y + 9 & 11x – y -23 = 0

b. Find the equation of lines parallel to $3x-4y-5=0$ at a unit distance from it.

Ans. Equation of any line parallel to 3x – 4y -5 = 0 is 3x – 4y + k = 0

Put x = 0 in 3x – 4y – 5 = 0, we get y = $\frac{-5}{4}$

Therefore, $\left(0, \frac{-5}{4}\right)$ is a point on the line 3x–4y-5 = 0 since the distance between the two lines is one unit therefore, the length of the perpendicular from $\left(0, \frac{-5}{4}\right)$ to 3x -4y + k = 0 is 1.

Hence $\dfrac{3\times 0 - 4\times\frac{-5}{4} + k}{\sqrt{3^2 + (-4)^2}} = \pm 1$

$\dfrac{5+k}{5} = \pm 1 \Rightarrow$ 5 + k = 5 and 5 + k = -5

$\Rightarrow$ k = 0 and k = -10

Putting the value of k = 0 and k = -10 in (i)

3x – 4y = 0 and 3x – 4y -10 = 0 are the required equation of the straight lines.

Ans $\begin{cases} 3x - 4y = 0 \\ 3x - 4y - 10 = 0 \end{cases}$

Q.9 a. Find the equation of the circle which passes through the points $(3,-2),(-2,0)$ and having its centre on the line $2x - y - 3 = 0$

Ans. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ________________(i)

as (i) passes through (3, -2)

$\therefore$ 9 + 4 + 6g - 4f + c = 0

or 13 + 6g – 4f + c = 0

or 6g – 4f + c = - 13____________(ii)

also (i) passes through (-2, 0)

$\therefore$ 4 + 0 – 4g – 0 + c = 0

or 4g – c = 4 ____________________(iii)

the centre (-g, -f) of (i) lies on 2x – y = 3

-2g + f = 3 _________________ (iv)

adding (ii) and (iii), we get

$10g - 4f = -9$ ________________ (v)

Solving (iv) and (v), we get,

$g = \frac{3}{2}$, $f = 6$

Putting g in (iii), we get, C = 2

Substituting these values of g, f and c in (i) we get,

$x^2 + y^2 + 3x + 12y + 2 = 0$

which is the required equation of circle.

$x^2 + y^2 + 3x + 12y + 2 = 0$

b. Find the vertex, focus directrix, latus-rectum and axis of parabola $3x^2 + 12x - 8y = 0$

Ans. The given equation is $3x^2 + 12x - 8y = 0$

$3(x^2 + 4x - \frac{8}{3}y) = 0$

or $x^2 + 4x + 4 - 4 - \frac{8}{3}y = 0$

or $(x + 2)^2 = \frac{8}{3}y + 4$

or $(x + 2)^2 = \frac{8}{3}(y + \frac{3}{2})$ ________________ (i)

Put $x + 2 = X$ and $y + \frac{3}{2} = Y$ (shifting the on origin)

$\therefore$ (i) reduces to $X^2 = \frac{8}{3}Y$ or $X^2 = 4\,(\frac{2}{3})\,Y$

Comparing with standard equation we get the following table:

Equation	$X^2 = \frac{8}{3}Y = 4(\frac{2}{3}Y)$	$(x+2)^2 = \frac{8}{3}(y+\frac{3}{2})$
Vertex	$(0, 0) \Rightarrow X = 0, Y = 0$	$\left(-2, \frac{3}{2}\right)$ [As $X = x + 2 \Rightarrow x = -2$ $Y = y + \frac{3}{2} \Rightarrow y = -\frac{3}{2}$]
Focus	$(0, \frac{2}{3}) \Rightarrow X=0, Y = \frac{2}{3}$	$(-2, \frac{-5}{6})$ [As $X = x + 2 \Rightarrow x = -2$ $Y = y + \frac{3}{2} = \frac{2}{3} \Rightarrow y = -\frac{5}{6}$]
Directrix	$Y + \frac{2}{3} = 0$	$Y + \frac{13}{6} = 0$ [As $y + \frac{3}{2} + \frac{2}{3} = 0$ $\Rightarrow y + \frac{13}{6} = 0$
L.R.	$\frac{8}{3}$	$\frac{8}{3}$
Axis	$X = 0$	$x + 2 = 0$

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